EXACT DIFFERENTIAL EQUATION

Exact Differential Equations are an important class of differential equation frequently encountered in Physics. Hence, it is important for us to know what exact differential equations are. We shall also explore a simple way of solving such equations.

Consider a 1st order linear differential equation

$$\frac{dy}{dx} = \frac{-A(x,y)}{B(x,y)}$$

$$\Rightarrow A(x,y)\,dx + B(x,y)\,dy = 0$$

How would you know this is an exact differential equation?

The given differential equation is an exact differential equation if

$$\boxed{\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}}$$

Test for exact D.E.

Remember,

$\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$ is just a condition to test whether any given differential equation is exact and can not be treated as a definition.

So, what would be our formal "definition" of an exact differential equation?

The equation $A(x,y)\,dx + B(x,y)\,dy = 0$

is an exact differential equation if there exists a function $\phi(x,y)$ such that $A(x,y)$ and $B(x,y)$ are separable as –

$$\boxed{A(x,y) = \frac{\partial \phi(x,y)}{\partial x}} \quad \text{and} \quad \boxed{B(x,y) = \frac{\partial \phi(x,y)}{\partial y}}$$

From this definition can we find out the solution of the differential equation?

$$A\,dx + B\,dy = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0$$

$$\Rightarrow d\phi = 0$$

$$\Rightarrow \phi = \text{const.}$$

$\phi(x,y) = \text{const.}$ is the solution to the differential equation.

Oh wait, can we derive the condition $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$ from our definition of an exact differential equation?

↪ Of course, we can!!

Let's do this –

From our definition,

$$A = \frac{\partial \phi}{\partial x} \qquad\qquad B = \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \frac{\partial A}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \qquad\qquad \Rightarrow \frac{\partial B}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

Since, $\dfrac{\partial^2 \phi}{\partial y \partial x} = \dfrac{\partial^2 \phi}{\partial x \partial y}$ so, $\boxed{\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}}$

Test for checking exact D.E.

Example : Check whether the differential equation $(4x^3 + 6xy + y^2)\dfrac{dx}{dy} = -(3x^2 + 2xy + 2)$ is an exact differential equation. Also, don't forget to solve it!

$$\frac{dy}{dx} = -\frac{4x^3 + 6xy + y^2}{3x^2 + 2xy + 2} = -\frac{P(x,y)}{Q(x,y)} \text{ (say)}.$$

$$\therefore \frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(4x^3 + 6xy + y^2) = 6x + 2y$$

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(3x^2 + 2xy + 2) = 6x + 2y$$

Since, $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$ so this differential equation is exact.

Let us construct the solution –

Let, $\phi(x,y) = C$ (const.) be the solution.

then, $\frac{\partial \phi}{\partial x} = P$

$$\Rightarrow \phi(x,y) = \int P\,dx + C_1(y)$$

$$= \int (4x^3 + 6xy + y^2)\,dx + C_1(y)$$

$$= x^4 + 3x^2y + xy^2 + C_1(y)$$

Again $\frac{\partial \phi}{\partial y} = Q$

$$\Rightarrow \phi(x,y) = \int (3x^2 + 2xy + 2)\,dy + C_2(x)$$

$$= 3x^2y + xy^2 + 2y + C_2(x)$$

Comparing both the expressions above

$$C_1(y) = 2y \quad ; \quad C_2(x) = x^4$$

Thus, the solution is –

$$\phi(x,y) = k$$

$$\Rightarrow x^4 + 3x^2y + xy^2 + 2y = k$$

We now consider another example -

$$2y\,dx + x\,dy = 0$$

$$\begin{array}{cc} \downarrow & \downarrow \\ A(x,y) & B(x,y) \end{array}$$

Let's check if the differential equation is exact.

$$\frac{\partial A}{\partial y} = 2 \quad ; \quad \frac{\partial B}{\partial x} = 1$$

Since, $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$ so this differential equation is not exact.

Now we multiply $\alpha(x,y) = x$ to the differential equation.

$$\alpha(x,y)\,2y\,dx + \alpha(x,y)\,x\,dy = 0$$

$$\Rightarrow x \cdot 2y\,dx + x \cdot x\,dy = 0$$

$$\Rightarrow 2xy\,dx + x^2 dy = 0$$

$$\begin{array}{cc} \downarrow & \downarrow \\ A & B \end{array}$$

Now is it an exact differential equation?

$$\frac{\partial A}{\partial y} = 2x \quad ; \quad \frac{\partial B}{\partial x} = 2x$$

Since, $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$ so this equation is indeed exact.

The takeaway lesson from the above exercise is that an inexact differential equation can be converted into an exact differential equation by multiplying with an appropriate function $\alpha(x,y)$, called the integrating factor.

Most of the time, this integrating factor is not easy to find.

However, there is a special class of differential equation where the integrating factor is not difficult to find.

Consider the equation :

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{P(x)y - Q(x)}{1}$$

$$\Rightarrow [P(x)y - Q(x)]\,dx + dy = 0$$

Let's say this differential equation is not exact but we can multiply an integrating factor $\alpha(x)$ (a function of x only) to make this differential equation an exact differential equation.

$$\underbrace{\alpha(x)\left[P(x)y - Q(x)\right]}_{A(x,y)}dx + \underbrace{\alpha(x)}_{B(x,y)}dy = 0$$

If this equation is an exact differential equation, then, there exists a function $\phi(x,y)$ such that

$$A(x,y) = \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \alpha(x)\left[P(x)y - Q(x)\right] = \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \quad \phi = \int \alpha(x)P(x)y\,dx - \int \alpha(x)Q(x)\,dx + C_1(y)$$

$$\Rightarrow \quad \phi = y\int \alpha(x)P(x)\,dx - \int \alpha(x)Q(x)\,dx + C_1(y)$$

$$B(x,y) = \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \alpha(x) = \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \phi = \alpha(x)\int dy + C_2(x)$$

$$= \alpha(x)y + C_2(x)$$

Comparing both expressions of ϕ we can say that

$$C_2(x) = -\int \alpha(x) Q(x)\, dx$$

$$C_1(y) = 0$$

and $\alpha(x) = \int \alpha(x) P(x)\, dx$

Now we would want to find the form of $\alpha(x)$

$$\Rightarrow \int \frac{d}{dx}(\alpha(x))\, dx = \int \alpha(x) P(x)\, dx$$

$$\Rightarrow \frac{d}{dx}(\alpha(x)) = \alpha(x) P(x)$$

$$\Rightarrow \frac{d(\alpha(x))}{\alpha(x)} = P(x)\, dx$$

Integrating both sides-

$$\int \frac{d(\alpha(x))}{\alpha(x)} = \int P(x)\, dx$$

$$\Rightarrow \ln(\alpha(x)) = \int P(x)\, dx$$

$$\Rightarrow \alpha(x) = e^{\int P(x)\, dx}$$

Now we can now construct the solution of the differential equation :

$$\phi(x, y) = \text{const.}$$

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$$\Rightarrow \alpha(x)y + C_2(x) = C$$

$$\Rightarrow \alpha(x)y - \int \alpha(x)\,Q(x)\,dx = C$$

$$\Rightarrow \alpha(x)y = \int \alpha(x)\,Q(x)\,dx + C$$

where $\alpha(x) = e^{\int P(x)dx}$

Thus, the solution to the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is $$\boxed{\alpha(x)y = \int \alpha(x)\,Q(x)\,dx + C}$$

where $\alpha(x) = e^{\int P(x)dx}$

↳ Aren't we already familiar with this!