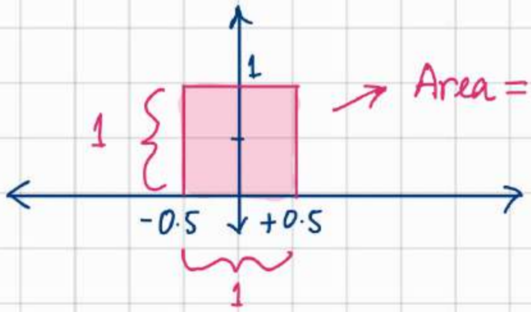
DIRAC DELTA FUNCTION

What is the Dirac Delta function ?

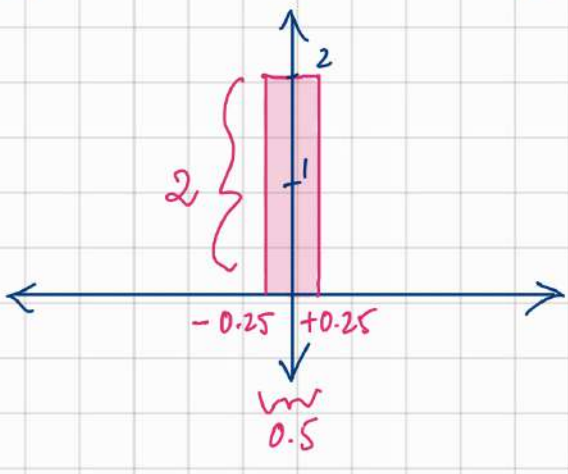
Let's say we have a square with length 1 and breadth 1 that rests on the x-axis.



The area of the square = length × breadth

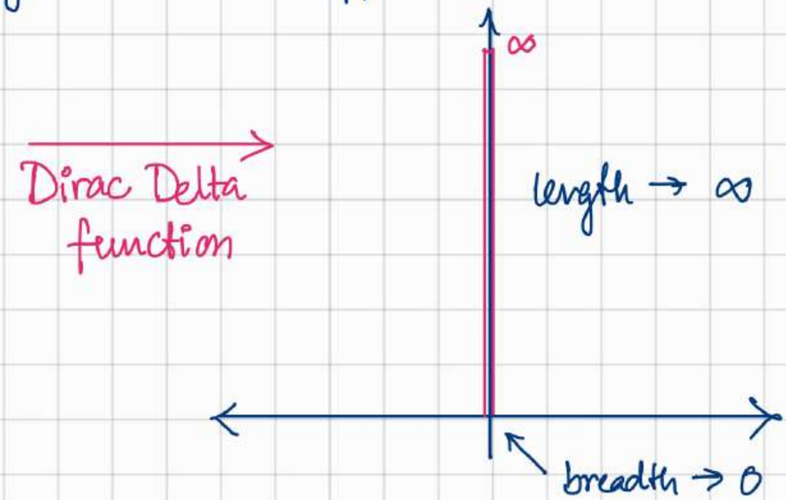
= 1 × 1 = 1

Suppose we compress the square from its sides keeping the area same i.e. 1. The length of the square will increase since the breadth decreases and we get a vertical rectangle.



Area of the rectangle = length × breadth
= 2 × 0.5 = 1

Now assume that we have compressed the rectangle keeping its area equal to 1 so much so that the breadth becomes effectively 0. In such a case the length shall approach ∞.



$$\text{Area} = \text{length} \times \text{breadth} = 1$$

This can be called a Dirac Delta function.

Although technically not a function, since the value of x is ∞ at 0, but

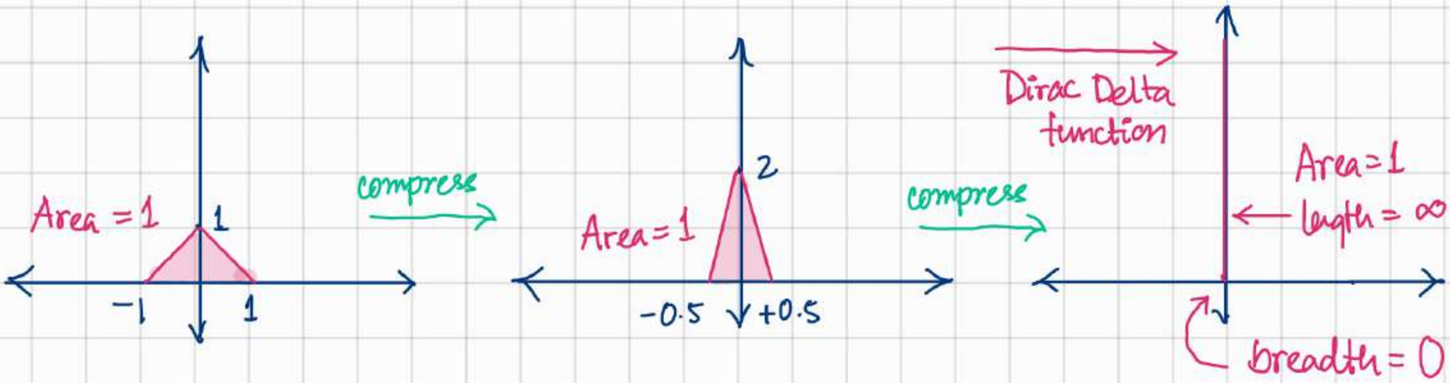
any function which is ∞ at a particular point in its domain and 0 everywhere else is called a Dirac Delta function.

i.e. $$\boxed{\delta(x) = \begin{cases} \infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}}$$ ** and area = 1

↳ definition of a Dirac Delta fuction

Although I illustrated the concept with a unit square, but you are free to use any function whose area under the curve is 1 to illustrate the Dirac Delta function concept.

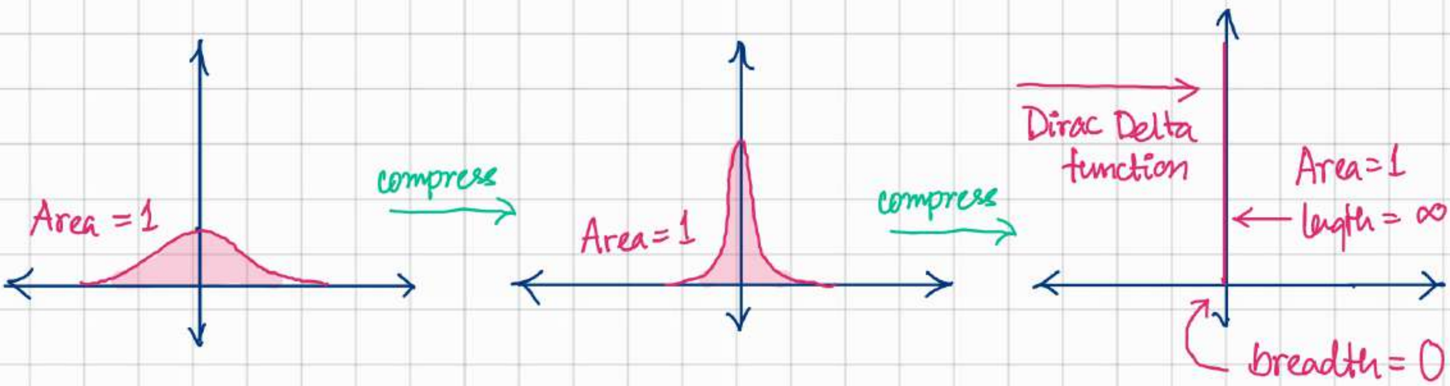
For eg: a unit triangle.



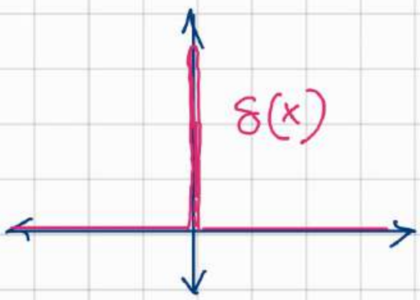
An important example : the Gaussian function

$$f(x) = e^{-x^2} \text{ (un-normalised)}$$

Point to remember : The area under the Gaussian curve is 1.

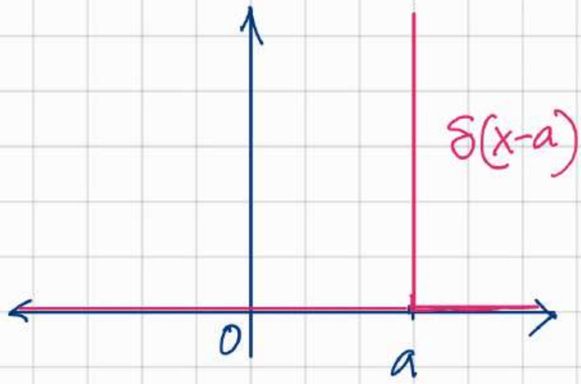


Thus, it is always convenient to imagine the Dirac Delta function $\delta(x)$ as a "function" that has a peak at $x=0$ and is 0 everywhere else

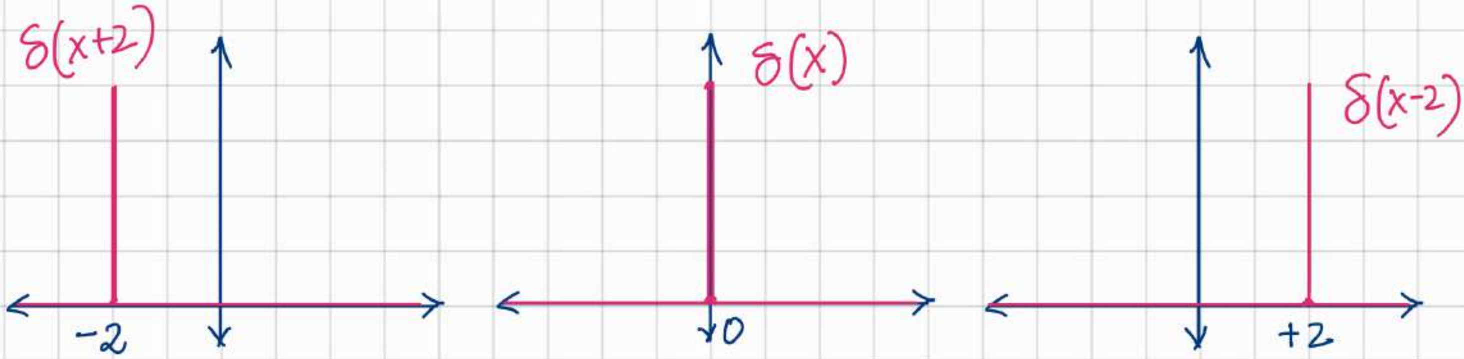


Let's say we do not want the delta function spike at $x = 0$, rather we want the spike at some other value of x. How can we write the Delta function for such a case.

We write the Delta function as $\delta(x-a)$ for the peak to be present at $x = a$



For example :-



$a = -2$ $a = 0$ $a = +2$

$\delta(x-(-2))$ or $\delta(x+2)$ $\delta(x-0)$ or $\delta(x)$ $\delta(x-2)$

Thus, we can modify our Delta function definition to accomodate this useful feature.

$$\boxed{\delta(x-a) = \begin{cases} \infty & \text{for } x = a \\ 0 & \text{for } x \neq a \end{cases}}$$ ** and area = 1

Definition of Dirac Delta function

Let's list out some of the important properties of the Dirac Delta Function –

Property 1 : (or rather a fact about the Delta function)

$$f(x)\,\delta(x-a) = f(a)\,\delta(x)$$

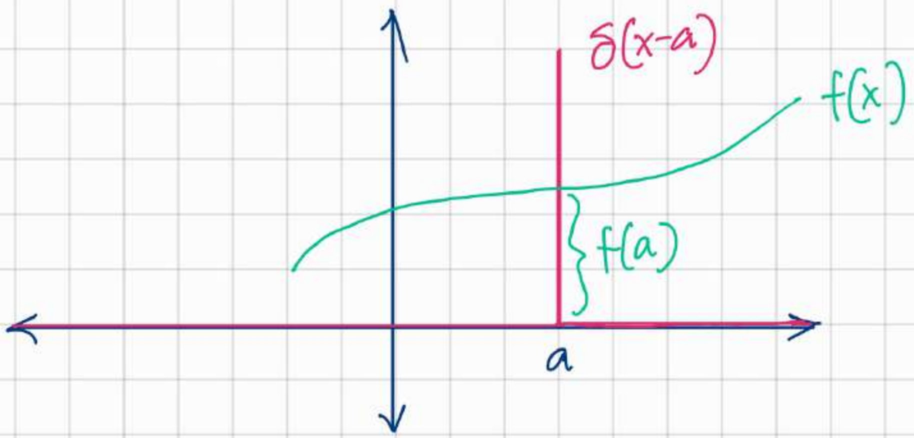
in particular, when $a=0$

$$f(x)\,\delta(x) = f(0)\,\delta(x).$$

Putting it into words, the product of a regular continuous function $f(x)$ with the Delta function $\delta(x-a)$ is just the product of the value of the function at a i.e. $f(a)$ with the Delta function $\delta(x-a)$.

Why is it so ?

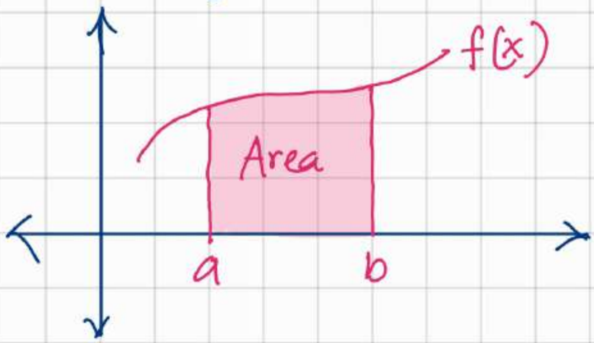
↳ Well, it is rather intuitive. Since the delta function is 0 everywhere except at $x=a$ so it won't matter if we replace $f(x)$ by $f(a)$.



$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

Property 2 : (or rather a defining property of the delta function)

Recall from your class 12 integral calculus what integration geometrically implies.



If you perform the definite integral

$$\int_a^b f(x)\,dx$$

it just gives the area under the curve of f(x) between the limits a and b.

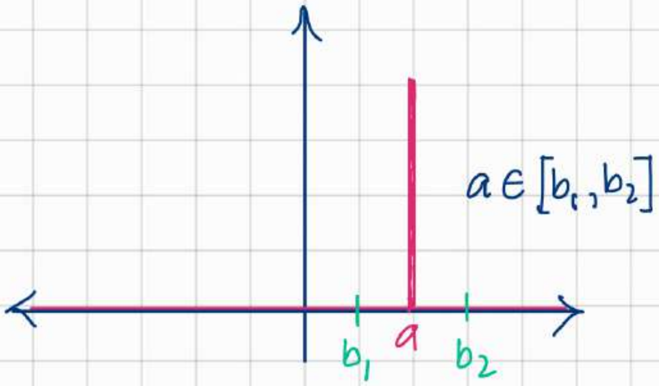
How did we develop the idea of Dirac Delta function ?

↳ We used unit area rectangles, unit area triangles and the Gaussian function which covers unit area under it.

So, if we integrate the Dirac Delta function within an interval that contains the peak, we should get 1.

i.e. $$\boxed{\int_{b_1}^{b_2} \delta(x-a)\,dx = 1}$$

if $a \in [b_1, b_2]$

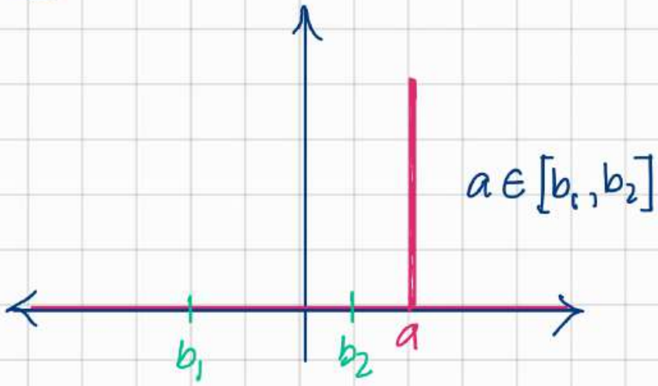


However, note that this integration will give the result 0 if a is not contained in the closed interval $[b_1, b_2]$

The area of 1 is entirely by the Dirac delta peak

$$\int_{b_1}^{b_2} \delta(x-a)\,dx = 0$$

if $a \notin [b_1, b_2]$



Always check the integration limits while performing integration using the Dirac Delta function.

In general, $$\boxed{\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1}$$

In fact, this property of the Dirac Delta function is so fundamental that we should include it in the definition of the Dirac Delta function.

We shall always use the following while defining the Dirac Delta Function -

(i) $$\delta(x-a) = \begin{cases} \infty & \text{for } x = a \\ 0 & \text{for } x \neq a \end{cases}$$

(ii) $$\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$$

Property 3 : (how we intend to use the delta function)

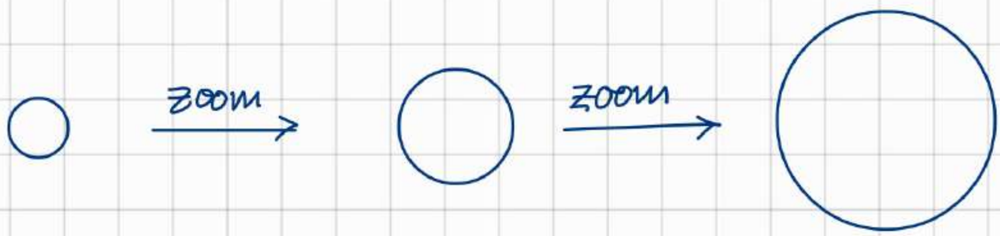
$\delta(x-a)$ is not a legitimate function mathematically since its value is ∞ at $x = a$.

but when used under the integral sign with another function it is tremendously useful.

We, as physicists, intend to use the delta function under the integral sign only.

"What is the use?" you ask?

Imagine, I draw a circle and zoom into it. What will happen to it? The circle gets bigger and bigger.



Now, let us "represent" a point with a dot. What happens when we "zoom" into it?

Well nothing happens since the point is a dimensionless object and the dot is just a representation of the point.

$$\bullet \xrightarrow{\text{zoom}} \bullet \xrightarrow{\text{zoom}} \bullet$$

In the electrostatics paper in the upcoming semester you are going to study about point charges.

Since, the dirac delta function is a distribution defined over all space, it can "pick out" such point charges and help us perform some useful calculations.

This is just a "trailer" to your upcoming semester course! More on it later.

Anyway let's get back to the mathematical property -

$$\boxed{\int_{b_1}^{b_2} f(x)\,\delta(x-a)\,dx = \begin{cases} f(a) & \text{if } a \in [b_1, b_2] \\ 0 & \text{otherwise} \end{cases}}$$

When used under the integral sign, the Dirac delta function $\delta(x-a)$ just picks out the value of the function at $x=a$ provided that a is contained within the integration limits. Otherwise it just gives 0 as the result.

In general, $$\boxed{\int_{-\infty}^{+\infty} f(x)\,\delta(x-a) = f(a)}$$

Property 4 :

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \frac{1}{|a|}\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = \frac{1}{|a|} f(0)$$

Proof : $$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx$$

Let, $ax = y$

$$\Rightarrow a\,dx = dy$$

$$\Rightarrow dx = \frac{1}{a}\,dy$$

When a is positive, $x \to \infty$ implies $y \to \infty$
and $x \to -\infty$ implies $y \to -\infty$

When a is negative, $x \to \infty$ implies $y \to -\infty$
$x \to -\infty$ implies $y \to +\infty$

So, if a is positive

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{+\infty} f(y/a)\,\delta(y)\cdot\frac{1}{a}\,dy$$

if a is negative

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \int_{+\infty}^{-\infty} f(y/a)\,\delta(y)\,\frac{1}{a}\,dy \longrightarrow \text{(i)}$$

To change the integration limits back to $(-\infty \rightarrow +\infty)$ we need to add a negative sign

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = -\int_{-\infty}^{+\infty} f(y/a)\,\delta(y)\,\frac{1}{a}\,dy$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{+\infty} f(y/a)\,\delta(y)\,\frac{1}{-a}\,dy \longrightarrow \text{(ii)}$$

To write (i) and (ii) under a single expression, we can use the definition of the modulus function.

$$|a| = \begin{cases} a & \text{for } a > 0 \\ -a & \text{for } a < 0 \end{cases}$$

Hence,

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{+\infty} f(y/a)\,\delta(y)\,\frac{1}{|a|}\,dy$$

$$= \frac{1}{|a|}\int_{-\infty}^{+\infty} f(y/a)\,\delta(y)\,dy$$

$$= \frac{1}{|a|}\,f(0/a)$$

$$= \frac{1}{|a|}\,f(0)$$

Thus, under the integral sign,

$\delta(ax)$ serves the same purpose as $\frac{1}{|a|}\,\delta(x)$.

$\hookrightarrow \quad \delta(ax) = \frac{1}{|a|}\,\delta(x)$

or $\boxed{\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \frac{1}{|a|}\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = \frac{1}{|a|}\,f(0)}$

—

In the particular case of $a = -1$

$$\boxed{\delta(-x) \equiv \delta(x)}$$

Property 5 : In fact we have the more general property for a function $g(x)$

$$\boxed{\delta\big(g(x)\big) = \sum_{x_0} \frac{\delta(x-x_0)}{|g'(x_0)|}}$$

here x_0 are the values where $x=0$

eg:- $$\delta(x^2-1) = \sum_{x_0=-1,+1} \frac{\delta(x-x_0)}{|2x_0|}$$

$$= \frac{\delta(x-1)}{|2|} + \frac{\delta(x+1)}{|-2|}$$

$$= \frac{\delta(x-1)+\delta(x+1)}{2}$$

CLASSWORK :-

Evaluate the following using the properties of the Dirac Delta function -

a) $\int_{2}^{6} (3x^2 - 2x - 1)\, \delta(x-3)\, dx$

b) $\int_{0}^{5} \cos x\, \delta(x-\pi)\, dx$

c) $\int_{0}^{3} x^3\, \delta(x+1)\, dx$

d) $\int_{-\infty}^{+\infty} \ln(x+3)\, \delta(x+2)\, dx$

e) $\int_{-2}^{2} (2x+3)\, \delta(3x)\, dx$

f) $\int_{0}^{2} (x^3 + 3x + 2)\, \delta(1-x)\, dx$

g) $\int_{-1}^{1} 9x^2\, \delta(3x+1)\, dx$

h) $\int_{-\infty}^{a} \delta(x-b)\, dx$