# GRAVITATION

Gravitation is one of the four types of fundamental forces, which exists in the form of attraction between any two massive objects in the universe and is directly proportional to the masses and inversely proportional to the square of the distance between the masses.

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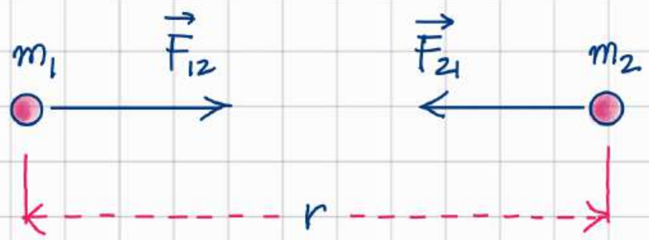
## Newton's Laws of Gravitation

* announced in the year 1687




The Law states that every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Mathematically, $F_G = G\dfrac{m_1 m_2}{r^2}$

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

## Universal Gravitational Constant, G

$$F_G = G \frac{m_1 m_2}{r^2}$$

If $m_1 = 1\ kg$ ; $m_2 = 1\ kg$
and $r = 1\ m$

then $F_G = G$

Thus, universal gravitational constant, G may be defined as the force of gravitational attraction between two unit masses separated by unit distance.

Question : Find the S.I. unit and dimension of G.

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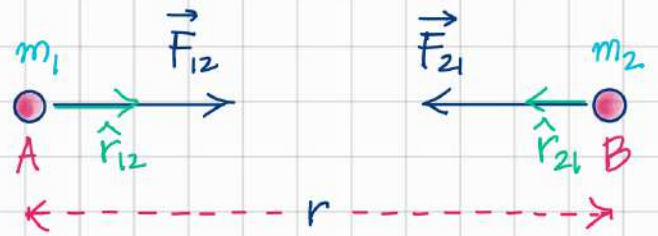
## Newton's Law of Gravitation in vector form




Consider two point masses of mass $m_1$ and $m_2$ located at points A and B respectively. The separation between the charges is r.

The force between the masses is always attractive.

Let $\vec{F}_{21}$ be the force exerted on $m_2$ by $m_1$

$\vec{F}_{12}$ be the force exerted on $m_1$ by $m_2$.

$\hat{r}_{21}$ be the unit vector along BA. $\left(\hat{r}_{21} = \frac{\vec{r}_{21}}{r}\right)$

$\hat{r}_{12}$ be the unit vector along AB.

then $$\vec{F}_{21} = G\frac{m_1 m_2}{r^2}\hat{r}_{21}$$

$$= G\frac{m_1 m_2}{r^2}\frac{\vec{r}_{21}}{r}$$

and $$\vec{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{r}_{12}$$

$$= G\frac{m_1 m_2}{r^2}\frac{\vec{r}_{12}}{r}$$

But we know that $\vec{r}_{21}$ and $\vec{r}_{12}$ are equal but opposite in direction.

$$\therefore \vec{r}_{21} = -\vec{r}_{12}$$

$$\Rightarrow \frac{\vec{r}_{21}}{r} = -\frac{\vec{r}_{12}}{r}$$

$$\Rightarrow \hat{r}_{21} = -\vec{r}_{12}$$

This gives us $\vec{F}_{21} = -\vec{F}_{12}$.

It tells us two important things :

1. the two masses exert equal and opposite force on each other. So, gravitational attraction obey Newton's third law of motion.

2. as gravitational force acts along the line joining the centres of two masses, so it is a central force.

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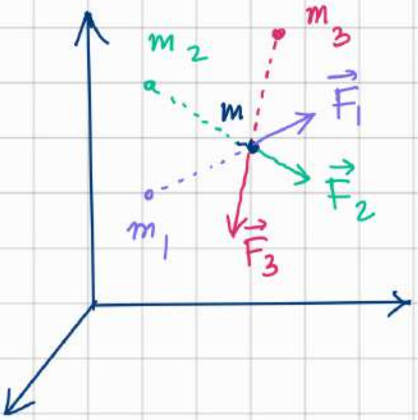
## Forces between multiple masses

If the system contains a number of interacting masses, then the force on a given mass is equal to the vector sum of the forces exerted on it by all remaining masses.




$$\boxed{\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3}$$

This is the superposition principle.

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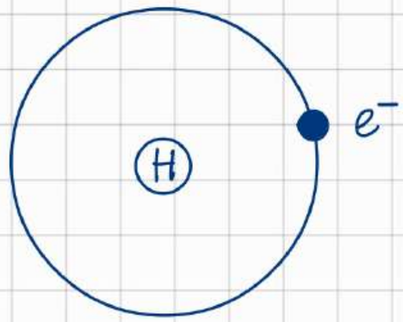
# Exercise : Comparison of Gravitational Force and Electromagnetic Force.

1.


Consider two bodies A and B, both of unit mass, one having charge of -1C and the other having a charge of +1C, seperated by a distance of 1m.

Which among the gravitational force and the electromagnetic force between the two bodies is stronger, and by how much?

2.




Consider an electron in the first Bohr orbit of an H-atom.

Which among the gravitational force and the electromagnetic force between the two bodies is stronger, and by how much?

## Gravitational Field :

The gravitational field due to massive body is the space around the body in which any other massive body is acted upon by a gravitational force of attraction.

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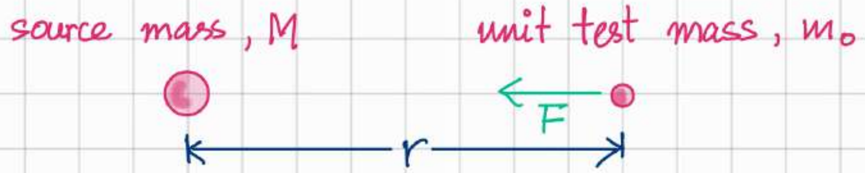
## Gravitational Field Strength

The gravitational field intensity (or strength) at a point in a gravitational field due to a source mass may be defined as the force experienced per unit test mass placed at that point without affecting the field of the source charge.




If $\vec{F}$ is the gravitational force of attraction on the unit test mass, $m_0$ then gravitational field strength at the point where $m_0$ is situated is

$$\boxed{\vec{E} = \frac{\vec{F}}{m_0}}$$

$$|\vec{E}| = \frac{1}{m_0} \times G\frac{Mm_0}{r^2}$$

$$\boxed{|\vec{E}| = G\frac{M}{r^2}}$$

The vector form for gravitational field strength or simply "field" is immensely helpful.


Suppose, the source mass is situated at the origin and the position vector of the test mass relative to the source mass be $\vec{r}$.

Since, the force of attraction is opposite to the direction of the position vector, $\vec{r}$

so the expression for field in vector form is –

$$\vec{E} = \frac{\vec{F}}{m_o}$$

$$\Rightarrow \vec{E} = \frac{1}{m_o}\left[G\frac{Mm_o}{r^2}(-\hat{r})\right]$$

$$\Rightarrow \boxed{\vec{E} = -G\frac{M}{r^2}\hat{r}}$$

$$\Rightarrow \vec{E} = -G\frac{M}{r^2}\cdot\frac{\vec{r}}{r}$$

$$\Rightarrow \boxed{\vec{E} = -\frac{GM}{r^3}\vec{r}}$$

S.I. unit – Newton / kg (N/kg)

It is a vector quantity.

Question : Find out its dimension.

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## Gravitational Potential

* Points to remember

(i) Gravitational field always has a negative sign, its highest value being 0 at $r = \infty$

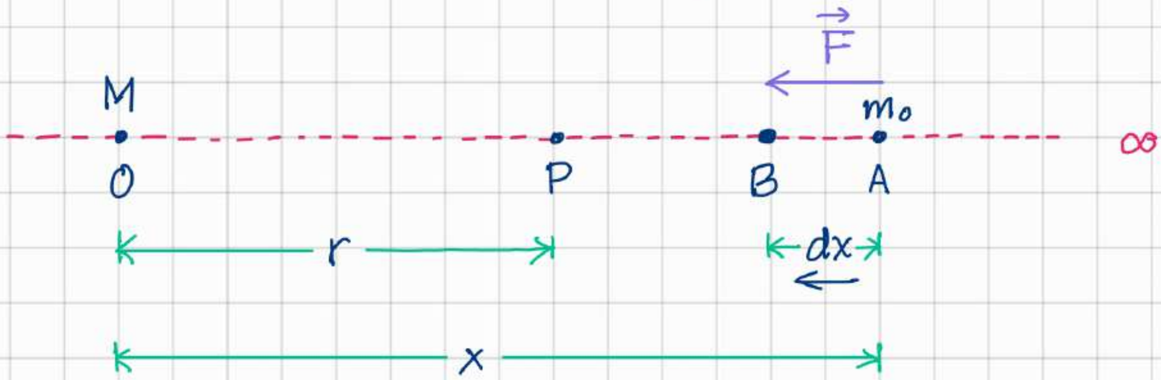
(ii) Gravitational potential also has a negative sign, its highest value being 0 at $r = \infty$.

The gravitational potential at a point in a gravitational field is equal to the work done in bringing a unit test mass from infinity to that point.

$$V = \frac{W}{m_0}$$

S.I. unit : Joule / kg.

# Gravitational Potential Due to a point mass




We consider a source mass M placed at O.

To calculate the potential due to the source mass at point P at a distance of r from it, let us consider a test mass $m_o$ placed at A at a distance of 'x' from O.

The force acting on test mass $m_o$ is given by

$$F = G\frac{Mm_o}{x^2}$$

The small amount of work done in moving the test mass $m_o$, from A to B through a small distance 'dx' by the gravitational field is given by –

$$dW = \bar{F}.d\bar{x}$$

$$= F\,dx\cos 0^\circ$$

$$= F\,dx$$

Therefore, the total work done in bringing the test mass $m_0$ from $\infty$ to the point P situated at a distance of 'r' from the origin is given by –

$$W = \int_{\infty}^{r} F\,dx$$

$$= \int_{\infty}^{r} G\frac{Mm_0}{x^2}\,dx$$

$$= GMm_0 \int_{\infty}^{r} \frac{1}{x^2}\,dx$$

$$= GMm_0 \left[-\frac{1}{x}\right]_{\infty}^{r}$$

$$= GMm_0 \left[-\frac{1}{r} + \frac{1}{\infty}\right]$$

$$= -G\frac{Mm_0}{r}$$

Thus, the net work done by the field on the test mass, $m_0$ is

$$W = -G\frac{Mm_0}{r}$$

Hence, the net potential at P is

$$V = \frac{W}{m_0} = \frac{1}{m_0} \cdot \left( - \frac{GMm_0}{r} \right) = - G\frac{M}{r}$$

$$\boxed{V = - \frac{GM}{r}}$$

## Convention about positive and negative sign.

Since, gravitational force is always attractive, hence the work done on the test mass by the field is negative.

So potential always has a negative value, the highest value being 0 at $r = \infty$.

Also the force always being attractive, the field always carries a negative sign.

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## Potential vs Potential Energy

Potential is defined for a unit positive test mass and is defined as work done by the field per unit mass on the test mass.

Its unit is Joule/kg

Whereas, potential energy has the actual unit of energy (ie Joules) and can be defined for any mass.

If V is the potential at any point in the gravitational field, then the potential energy for any mass m at that point is -

$$P.E. = Vm$$

↳ unit : Joule/kg × kg = Joule

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Important relationship between field and potential

Field is the negative gradient of potential.

Consider the simplest case of one-dimension,

$$\bar{E} = -\bar{\nabla} V$$

$$= -\frac{\partial}{\partial r}\left(-G\frac{M}{r}\hat{r}\right)$$

$$= +\left[GM\left(-\frac{1}{r^2}\right)\hat{r}\right] = -\frac{GM}{r^2}\hat{r}$$

Illustration :-


Consider two points A and B in the gravitational field for source mass M.

Potential difference between the points A and B is the work done in bringing test mass $m_0$ from A to B.

$$\therefore \quad \Delta V = \frac{\Delta W}{m_0}$$

$$\Rightarrow \Delta V = \frac{\vec{F} \cdot \Delta \vec{r}}{m_0}$$

$$\Rightarrow \Delta V = \frac{F \Delta r \cos 180^\circ}{m_0}$$

$$\Rightarrow \Delta V = -\frac{F \Delta r}{m_0}$$

$$\Rightarrow \frac{F}{m_0} = -\frac{\Delta V}{\Delta r}$$

$$\Rightarrow E = -\frac{\Delta V}{\Delta r}$$

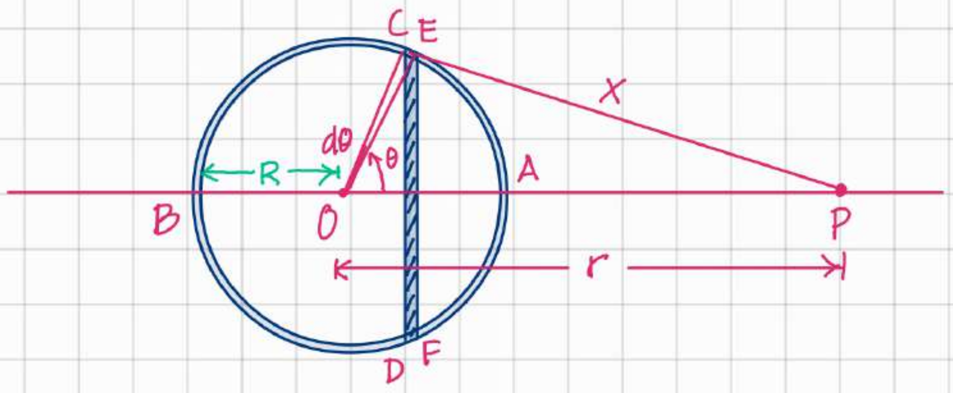
In the limit $\Delta r \rightarrow 0$

$$\boxed{E = -\frac{dV}{dr}}$$

$\frac{dV}{dr}$ is called the potential gradient

Potential gradient is the rate at which potential changes with distance.

Question : Find the potential and field inside and outside a massive spherical shell.




Suppose the total mass of the sphere is $M$. and the total surface area is $S$.

$\sigma = \frac{M}{S}$ is the surface mass density or mass per unit area.

We want to find the field at the point P which is situated at a distance of r from the center of the spherical shell of radius R.

We cut out a slice CEFD in the form of a ring by two planes CD and EF close to each other and perpendicular to the radius OA of the shell.

Let $\angle EOP = \theta$ and $\angle COE = d\theta$

Radius of the ring, $EK = OE \sin\theta = R\sin\theta$

So, circumference of the ring $= 2\pi R \sin\theta$.

Area of the ring = circumference x width

$$= 2\pi R \sin\theta \, . \, R\,d\theta$$

$$= 2\pi R^2 \sin\theta \, d\theta$$

Mass of the ring = area of the ring x surface mass density.

$$= 2\pi R^2 \sin\theta \, d\theta \, . \, \sigma$$

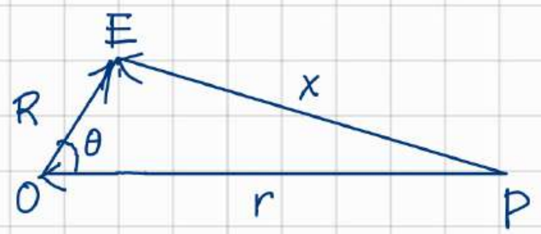
$$= 2\pi R^2 \sigma \sin\theta \, d\theta$$

If $EP = x$, then every point on the ring is at a distance of $x$ from P.

Thus, potential (being a scalar quantity) at P due to the ring is

$$dV = -G \frac{\text{mass of ring}}{x}$$

$$= -G \frac{2\pi R^2 \sigma \sin\theta \, d\theta}{x}$$




Now consider the triangle OEP.

$$\vec{OP} + \vec{OE} = \vec{PE}$$

$$\Rightarrow \quad PE^2 = OP^2 + OE^2 - 2OP.OE\cos\theta$$

$$\Rightarrow \quad x^2 = r^2 + R^2 - 2Rr\cos\theta$$

Differentiating,

$$2x\,dx = 0 + 0 + 2Rr\sin\theta\,d\theta$$

$$\Rightarrow \quad x = \frac{Rr\sin\theta\,d\theta}{dx}$$

Substituting the value of x in the expression for dV,

$$dV = -G\frac{2\pi R^2\sigma\sin\theta\,d\theta}{x}$$

$$= -G\frac{2\pi R^2\sigma\sin\theta\,d\theta}{Rr\sin\theta\,d\theta}\,dx$$

$$= -G\frac{2\pi R\sigma}{r}\,dx$$

To get the total potential V, we need to integrate this expression within the limits $x = AP = (r-R)$ to $x = BP = (r+R)$

$$\therefore \quad V = \int_{(r-R)}^{(r+R)} -G\frac{2\pi R\sigma}{r}\,dx$$

$$\Rightarrow V = -G\frac{2\pi R\sigma}{r}\int_{r-R}^{r+R} dx$$

$$= -G\frac{2\pi R\sigma}{r}\left[x\right]_{r-R}^{r+R}$$

$$= -G\frac{2\pi R\sigma}{r}\left[r+R-r+R\right]$$

$$= -G\frac{4\pi R^2\sigma}{r}$$

$$= -G\frac{S\sigma}{r}$$

$$\boxed{V = -G\frac{M}{r}}$$

The gravitational potential at P due to the whole shell is $V = -\frac{GM}{r}$ which is the same as that due to a point mass M at O.

In other words,

for a point outside the surface of the shell, the mass of the shell behaves as though it were concentrated at its centre.

It is really easy to find the field if we know the potential.

Field is just the negative gradient of field.

$$E = -\bar{\nabla} V$$

$$= -\frac{\partial}{\partial r}\left(-G\frac{M}{r}\right)\hat{r}$$

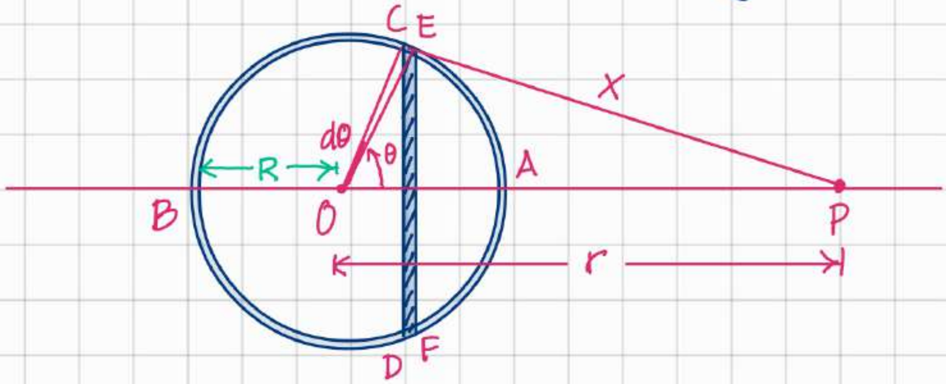
$$= +GM\left(-\frac{1}{r^2}\right)\hat{r}$$

$$\boxed{E = -G\frac{M}{r^2}\hat{r}}$$

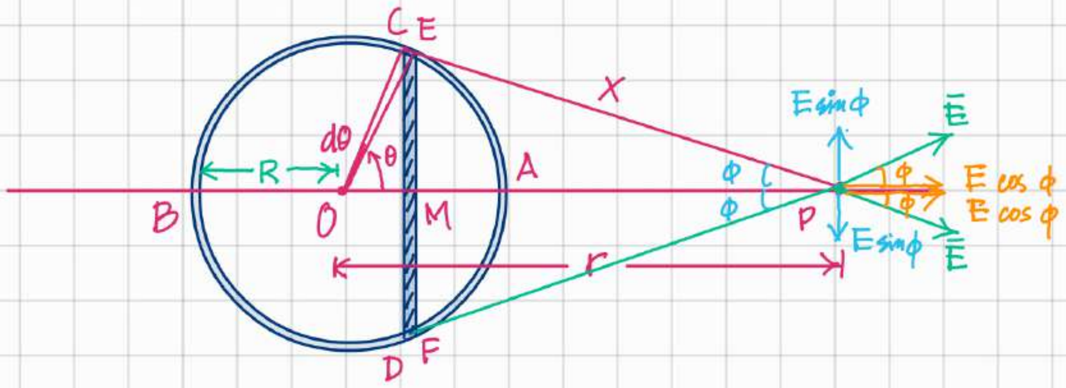
which is the same if the whole mass were concentrated at its centre.

Well, is there a way of calculating the field without calculating the potential first?

↳ Of course there is. Let's try this –



But the field being a vector quantity, we need to carefully resolve the resultant field at P into its components.




As evident from the figure, the sine-component of the field at P is cancelled out by diametrically opposite mass elements on the ring. Only the cos-component of E should be taken for calculation of field at point P.

$$\text{Also note that, } \cos\phi = \frac{PM}{PE} = \frac{OP-OM}{PE}$$

$$= \frac{r - R\cos\theta}{x}$$

Field at E due to the ring,

$$dE = -G\,\frac{\text{mass of ring}}{x^2}\cos\phi$$

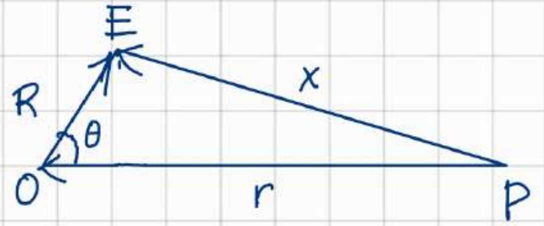
$$= -G\,\frac{2\pi R^2\sigma\sin\theta\, d\theta}{x^2}\cdot\left(\frac{r-R\cos\theta}{x}\right)$$

$$= -G\,\frac{2\pi R^2\sigma}{x^3}\,\frac{1}{2r}\left(2r^2 - 2rR\cos\theta\right)\sin\theta\, d\theta$$

$$= -G\,\frac{\pi R^2 \sigma}{r x^3}\left(-R^2 + r^2 + \underline{r^2 + R^2 - 2rR\cos\theta}\right)\sin\theta\, d\theta$$

$$= -G\,\frac{\pi R^2 \sigma}{r x^2}\,\frac{1}{x}\left(r^2 - R^2 + x^2\right)\sin\theta\, d\theta$$

We would ideally want to perform this integration using $x$ as the variable.

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Differentiating,

$$2x\,dx = 0 + 0 + 2Rr\sin\theta\, d\theta$$

$$\Rightarrow \sin\theta\, d\theta = \frac{2x\,dx}{2Rr}$$

$$\Rightarrow \sin\theta\, d\theta = \frac{x\,dx}{Rr}$$

$$\therefore \quad dE = -G\frac{\pi R^2\sigma}{rx^2}\frac{1}{x}\left(r^2 - R^2 + x^2\right)\sin\theta d\theta$$

$$= -G\frac{\pi R^2\sigma}{rx^2}\frac{dx}{rR\sin\theta d\theta}\left(r^2 - R^2 + x^2\right)\sin\theta d\theta$$

$$= -G\frac{\pi R\sigma}{r^2}\left[\frac{r^2 - R^2}{x^2}dx + dx\right]$$

To get the total field E, we need to integrate this expression within the limits $x = AP = (r-R)$ to $x = BP = (r+R)$

$$E = -G\frac{\pi R\sigma}{r^2}\left[\int_{r-R}^{r+R}\frac{r^2 - R^2}{x^2}dx + \int_{r-R}^{r+R}dx\right]$$

$$= -G\frac{\pi R\sigma}{r^2}\left[\left(r^2 - R^2\right)\left[-\frac{1}{x}\right]_{r-R}^{r+R} + \left[x\right]_{r-R}^{r+R}\right]$$

$$= -G\frac{\pi R\sigma}{r^2}\left[\left(r^2 - R^2\right)\left(-\frac{1}{r+R} + \frac{1}{r-R}\right) + r + R - r + R\right]$$

$$= -G\frac{\pi R\sigma}{r^2}\left[\left( -(r-R) + (r+R) + 2R\right]$$

$$= -\frac{G\pi R\sigma}{r^2}\left[4R\right]$$

$$= -G\frac{4\pi R^2\sigma}{r^2} = -G\frac{S\sigma}{r^2} = -G\frac{M}{r^2}$$

$$\boxed{E = -G\frac{M}{r2}\hat{r}}$$

We obtained similar result using this technique.

Next we want to obtain the potential and field at a point on the surface of the shell.

On the surface, $r = R$.

So we may simply replace r with R in the expressions for potential and field at a point outside the spherical shell.

Potential at a point outside the shell,

$$V = -\frac{GM}{r}$$

Hence, potential at a point on the shell surface

$$\boxed{V = -\frac{GM}{R}}$$

On the other hand, field at a point outside the shell

$$E = -\frac{GM}{r^2}\hat{r}$$

Hence, field at a point on the surface of the shell

$$\boxed{E = -\frac{GM}{R^2}\hat{r}}$$

Warning !!

As a newbie, you might be tempted to take the negative gradient of the potential at the surface to find the field.

$$E = -\bar{\nabla}\left(-G\frac{M}{R}\right) = -\frac{\partial}{\partial r}\left(-G\frac{M}{R}\right) = 0 \quad \times$$

But this is wrong.

$-G\frac{M}{R}$ is the value of the potential at the surface. You are bound to get 0 if you take its gradient.

The expression for potential is in fact $-\frac{GM}{r}$.

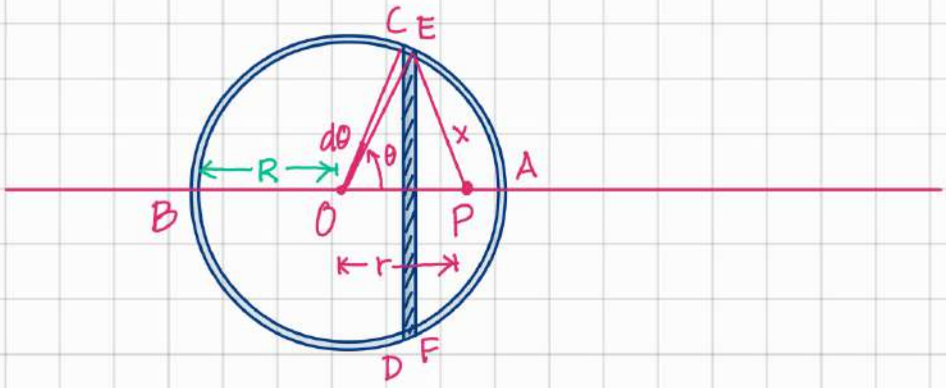
So, the correct way to do this is to take the negative gradient of this expression and then plug in the value $r = R$.

$$E = -\bar{\nabla}\left(-G\frac{M}{r}\right)\Bigg|_{r=R} = -\frac{GM}{r^2}\hat{r}\Bigg|_{r=R} = -\frac{GM}{R^2}\hat{r}$$

Next we want to find the potential and field inside the spherical shell.

The technique remains same, only the integration limits will change in the expression

$$V = \int_a^b - G \frac{2\pi R\sigma}{r} dx$$




To get the total potential V, we need to integrate this expression within the limits $x = AP = (R-r)$ to $x = BP = (R+r)$

$$\therefore \quad V = \int_{(R-r)}^{R+r} - G \frac{2\pi R\sigma}{r} dx$$

$$= - G \frac{2\pi R\sigma}{r} \left[ x \right]_{R-r}^{R+r}$$

$$= - G \frac{2\pi R\sigma}{r} \left( R+r-R+r \right)$$

$$V = -G\frac{2\pi R\sigma}{r}(2r)$$

$$= -G4\pi R\sigma$$

$$= -G\frac{4\pi R^2\sigma}{R}$$

$$\boxed{V = -G\frac{M}{R}}$$

inside the shell

Thus, potential is constant everywhere inside the spherical shell.

Field inside the spherical shell is 0 since the negative gradient of a constant potential will give us 0.

$$\bar{E} = -\bar{\nabla}V = -\frac{\partial}{\partial r}\left(-G\frac{M}{R}\hat{r}\right) = 0$$

$$\boxed{\bar{E} = 0}$$

Let's convince ourselves by performing the actual integral of the expression

$$dE = -G\frac{\pi R\sigma}{r^2}\left[\frac{r^2-R^2}{x^2}dx + dx\right]$$

To get the total field E, we need to integrate this expression within the limits $x = AP = (R-r)$ to $x = BP = (R+r)$

$$E = -G\frac{\pi R\sigma}{r^2}\left[\int_{R-r}^{R+r}\frac{r^2-R^2}{x^2}\,dx + \int_{R-r}^{R+r}dx\right]$$

$$= -G\frac{\pi R\sigma}{r^2}\left[(r^2-R^2)\left[-\frac{1}{x}\right]_{R-r}^{R+r} + \left[x\right]_{R-r}^{R+r}\right]$$

$$= -G\frac{\pi R\sigma}{r^2}\left[(r^2-R^2)\left(-\frac{1}{R+r}+\frac{1}{R-r}\right)+R+r-R+r\right]$$

$$= -G\frac{\pi R\sigma}{r^2}\left[(r^2-R^2)\frac{(-(R-r)+(R+r))}{(R^2-r^2)}+2r\right]$$

$$= -\frac{G\pi R\sigma}{r^2}\left[-2r+2r\right]$$

$$= 0$$

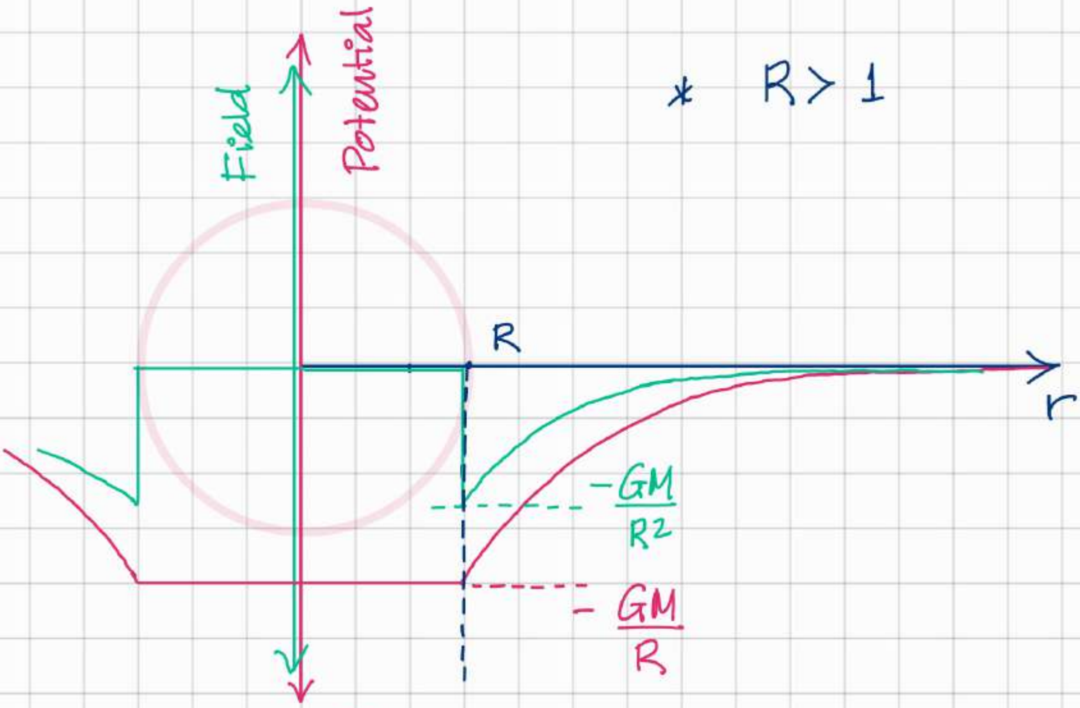
This is an important fact we discovered!

The field (and hence force) inside a massive uniform spherical shell is zero.

Summary..

| | Potential | Field |
|---|---|---|
| Outside the shell | $-G\frac{M}{r}$ | $-G\frac{M}{r^2}\hat{r}$ |
| on the shell surface | $-G\frac{M}{R}$ | $-G\frac{M}{R^2}\hat{r}$ |
| inside the shell | $-G\frac{M}{R}$ | $0$ |

* $R > 1$




* $R < 1$