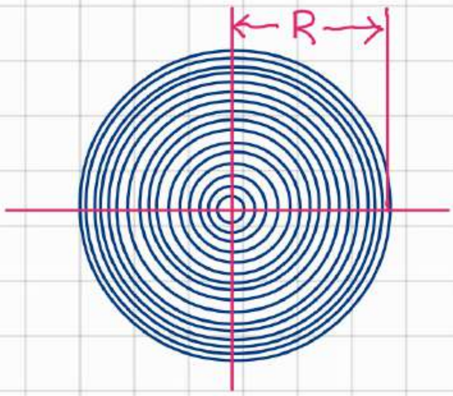
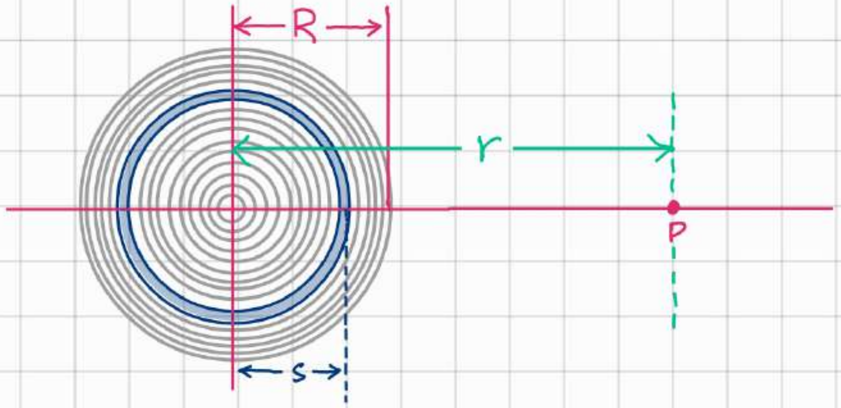
Question : Find the potential and field inside and outside a massive solid sphere of uniform mass density.

We can do a standalone derivation for finding out the expression for potential and field for a solid sphere but let's capitalize on our knowledge gained from the treatment of a spherical shell to find out the potential and field.



A solid sphere can be thought of as a collection of uniform shells of concentric centres with radius ranging from 0 to R.



To find the potential at P due to the whole sphere we shall find the potential due to one arbitrary shell with radius s and subsequently integrate the expression within proper limits.

Let us suppose the sphere has a uniform mass density of ρ.

ie. $\rho = \frac{M}{vol}$ (M is the total mass of the solid sphere)

Suppose the thickness of the arbitrary shell is ds

Hence, volume of the spherical shell is

$$vol = \text{surface area} \times \text{thickness}$$

$$= 4\pi s^2 ds$$

Hence, mass of the shell is $\underline{m} = 4\pi \rho s^2 ds$

Potential at P (which is outside the solid sphere) due to this spherical shell is

$$dV = -G\frac{m}{r}$$

$$= -G\frac{4\pi\rho s^2 ds}{r}$$

To get the total potential at point P we have to integrate the above expression within the integration limits $s = 0$ to $s = R$.

$$V = \int_0^R \left(-G\frac{4\pi\rho s^2 ds}{r}\right)$$

$$\Rightarrow V = -G\frac{4\pi\rho}{r}\int_0^R s^2\,ds$$

$$= -G\frac{4\pi\rho}{r}\left[\frac{s^3}{3}\right]_0^R$$

$$= -G\,\frac{4}{3}\pi R^3\rho\,\frac{1}{r}$$

$$\boxed{V = -G\frac{M}{r}}$$

Taking the negative gradient of the potential gives the field at the point.

$$\vec{E} = -\vec{\nabla}V$$

$$= -\frac{\partial}{\partial r}\left(-G\frac{M}{r}\hat{r}\right)$$

$$\boxed{\vec{E} = -\frac{GM}{r^2}\hat{r}}$$

The gravitational potential at P due to the whole solid sphere is $V = -\dfrac{GM}{r}$ which is the same as that due to a point mass M at O.

In other words,
for a point outside the solid sphere the mass of the sphere behaves as though it were concentrated at its centre.

Finding the potential and field at the surface of the solid sphere is not at all difficult.

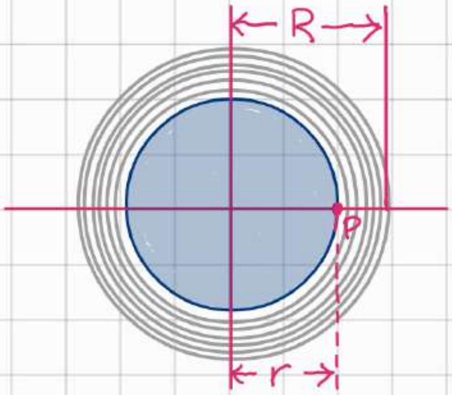
We shall plug in $r = R$ in the expressions obtained above.

On the surface,

potential, $\boxed{V = -\dfrac{GM}{R}}$

field, $\boxed{\bar{E} = -\dfrac{GM}{R^2}\hat{r}}$

Next we want to find the value of potential and field at a point inside the sphere.



The potential at the point P has contribution from two parts of the sphere, one part consisting of concentric spherical shells with radius greater than r and the other part consisting of a solid sphere of radius, r.

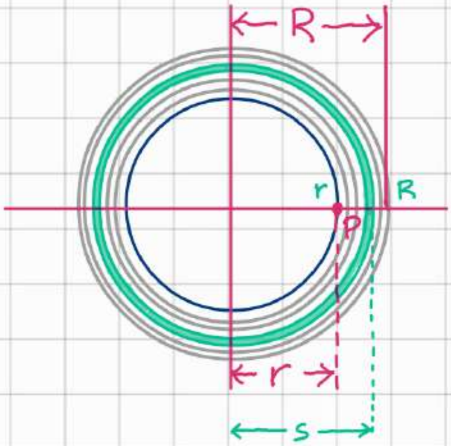
For the inner solid sphere of radius 'r', the potential at distance r is -

$$V_1 = -G\,\frac{\text{mass of sphere}}{r}$$

$$= -G\ \frac{4}{3}\pi r^3 \rho G \cdot \frac{1}{r}$$

$$= -\frac{4}{3}\pi r^2 \rho G$$

For the part consisting of concentric shells with radius greater than r, we have to integrate the expression within the limits s=r to s=R.



$$V_2 = -G\int_r^R \frac{4\pi\rho s^2 ds}{s}$$

$$= -G\,4\pi\rho \int_r^R s\,ds$$

$$= -G4\pi\rho \cdot \left[\frac{s^2}{2}\right]_r^R$$

$$= -G4\pi\rho\left(\frac{R^2-r^2}{2}\right)$$

Thus, potential due to the whole sphere

$$V = V_1 + V_2$$

$$\Rightarrow V = -G\frac{4}{3}\pi r^2\rho - G4\pi\rho\left(\frac{R^2-r^2}{2}\right)$$

$$\Rightarrow V = -G\frac{4}{3}\pi\rho\left[r^2 + 3\left(\frac{R^2-r^2}{2}\right)\right]$$

$$\Rightarrow V = -G\frac{4}{3}\pi R^3\rho\left[\frac{2r^2+3R^2-3r^2}{2R^3}\right]$$

$$\Rightarrow \boxed{V = -GM\left[\frac{3R^2-r^2}{2R^3}\right]}$$

Question : What is the potential at the centre of the solid sphere ?

$$V_0 = -GM\left[\frac{3R^2-0^2}{2R^3}\right]$$

$$= -\frac{3}{2}\frac{GM}{R}$$

maximum value

Thus, potential at the centre is $\frac{3}{2}$ times the potential at the surface of the solid sphere.

Taking the negative gradient of the potential gives the field at the point.

$$\bar{E} = -\bar{\nabla} V$$

$$\Rightarrow \bar{E} = -\frac{\partial}{\partial r}\left(-GM\left[\frac{3R^2 - r^2}{2R^3}\right]\right)\hat{r}$$

$$\Rightarrow \vec{E} = \frac{GM}{2R^3}(-2r)\,\hat{r}$$

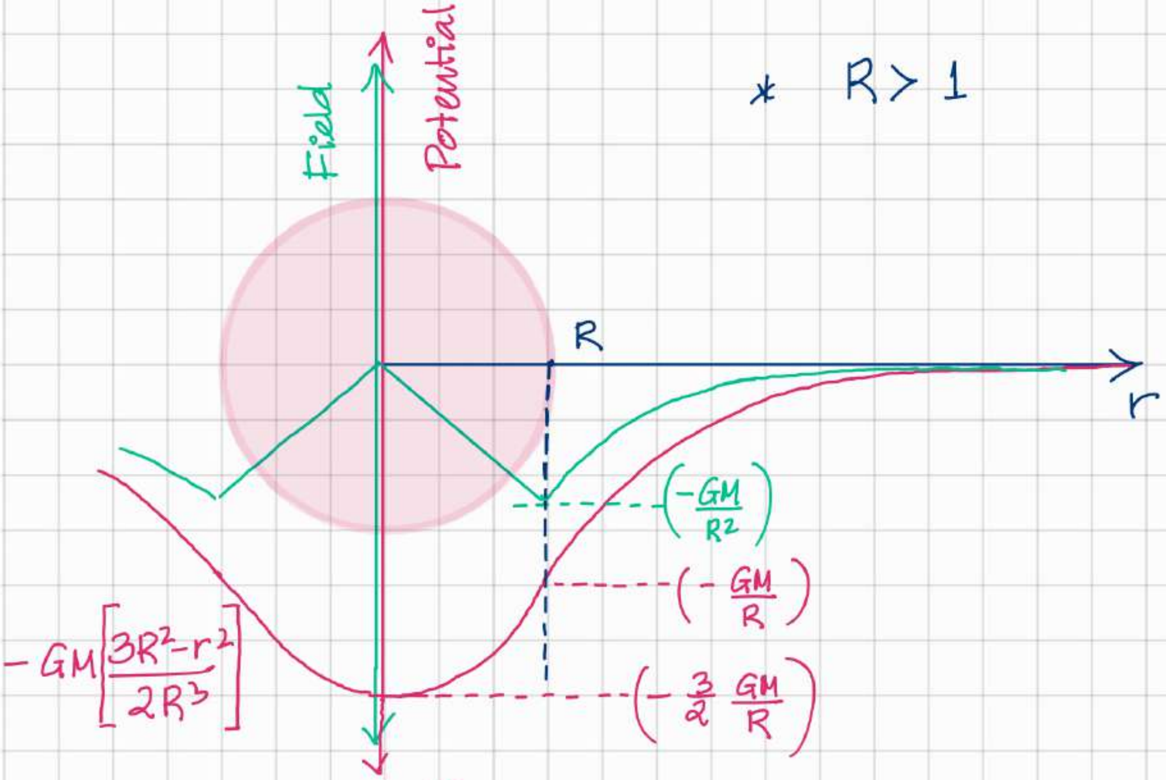
$$\Rightarrow \vec{E} = -G\frac{M}{R^3} r\,\hat{r} \qquad (\text{i.e. } E \propto r)$$

Thus, the gravitational field at a point inside a massive solid sphere is directly proportional to the distance of the point from the centre of the sphere.

Summary..

	Potential	Field
outside the sphere	$-G\frac{M}{r}$	$-G\frac{M}{r^2}\hat{r}$
on the sphere's surface	$-G\frac{M}{R}$	$-G\frac{M}{R^2}\hat{r}$
inside the sphere	$-GM\left[\frac{3R^2 - r^2}{2R^3}\right]$	$-G\frac{M}{R^3}r\,\hat{r}$

$*\quad R > 1$



$*\quad R < 1$