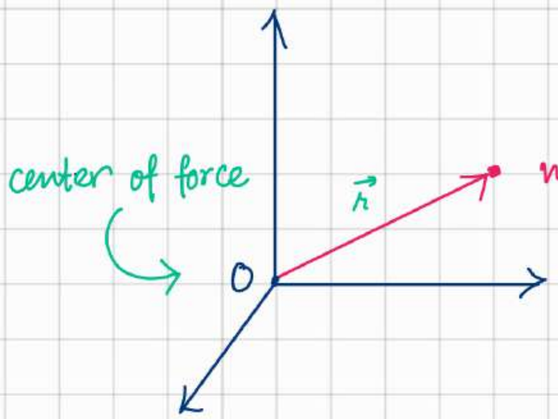
Motion of a particle under a central force field

Central forces – Gravitational force
Coulombic / Electromagnetic force

So, what is a central force ?



Suppose, O is the center of force (for eg, consider the source of a gravitational field)

Definition :- The force acting on the particle of mass m is called a central force if -

- the force is always directed from m toward, or away, from the center of force O.
- the magnitude of the force only depends on the distance r from O.

Mathematically, F is a central force if and only if

$$\vec{F} = f(r)\,\hat{r} = f(r)\,\frac{\vec{r}}{r}$$

Properties of a particle moving under the influence of a central force

1. The path or orbit of the particle must be a plane curve i.e. the movement of the particle is restricted to a plane.

2. The angular momentum of the particle is conserved.

3. Law of areas : The areal velocity of the particle is constant. In other words, the time rate of change of area is constant.

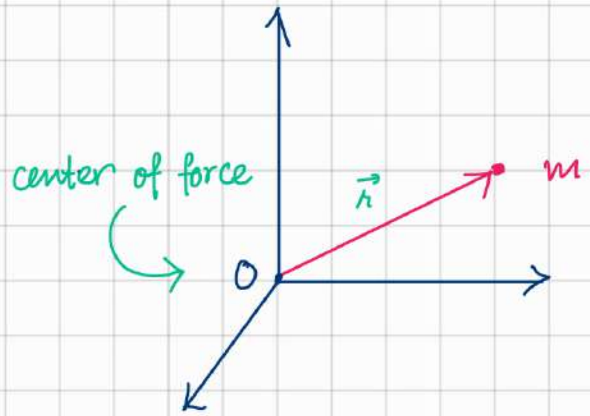
   [the particle moves in such a way that the position vector or the radius vector drawn from O to the particle sweeps out equal areas in equal time.]

4. A central force is conservative in nature.

Property 1 :

The path or orbit of the particle must be a plane curve i.e. the movement of the particle is restricted to a plane.

Proof :-



Let $\vec{F} = f(r)\hat{r}$ be the central force field.

We take the cross-product of position vector $\vec{r}$ and $\vec{F}$

$$\vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r}$$

$$= f(r)\left[\vec{r} \times \hat{r}\right]$$

Since $\vec{r}$ as $\hat{r}$ are in the same direction
Therefore,

$$\vec{r} \times \hat{r} = 0$$

This implies $\vec{r} \times \vec{F} = 0 \longrightarrow (1)$

The force $\vec{F}$ may be written as $\vec{F} = m\dfrac{d\vec{v}}{dt}$

Therefore (1) implies, $\vec{r} \times m\dfrac{d\vec{v}}{dt} = 0$

$$\Rightarrow m\left(\vec{r} \times \frac{d\vec{v}}{dt}\right) = 0$$

$$\Rightarrow \vec{r} \times \frac{d\vec{v}}{dt} = 0 \longrightarrow (2)$$

Now $\quad \frac{d}{dt}(\vec{r} \times \vec{v})$

$$= \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= \vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= 0 + \vec{r} \times \frac{d\vec{v}}{dt}$$

Thus, $\quad \vec{r} \times \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{v})$

Substituting in (2),

$$\vec{r} \times \frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$

$$\Rightarrow \vec{r} \times \vec{v} = \vec{h} \quad \text{(a constant vector in time)} \longrightarrow (3)$$

$$\Rightarrow \vec{r} \cdot (\vec{r} \times \vec{v}) = \vec{r} \cdot \vec{h}$$

We may use the property of vector -

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

to write $\quad (\vec{r} \times \vec{r}) \cdot \vec{v} = \vec{r} \cdot \vec{h}$

$$\Rightarrow 0 = \vec{r} \cdot \vec{h}$$

Since $\vec{r} \cdot \vec{h} = 0$, so $\vec{h}$ is a vector perpendicular to $\vec{r}$.

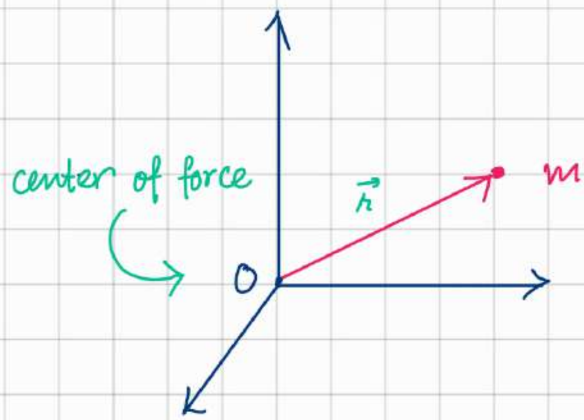
Since, $\vec{h} = \vec{r} \times \vec{v}$ is a vector that is constant in time i.e. the direction of $\vec{h}$ is fixed and do not change in time, so $\vec{r}$ and $\vec{v}$ must be confined to a plane.

Thus, the motion of a particle in a central force field is restricted to a plane.

proved.

Property 2 : The angular momentum of the particle is conserved.

Proof :



Let $\vec{F} = f(r)\hat{r}$ be the central force field.

We take the cross-product of position vector $\vec{r}$ and $\vec{F}$

$$\vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r}$$

$$= f(r)\left[\vec{r} \times \hat{r}\right]$$

Since $\vec{r}$ & $\hat{r}$ are in the same direction

Therefore,

$$\vec{r} \times \hat{r} = 0$$

This implies $\vec{r} \times \vec{F} = 0 \longrightarrow (1)$

The force $\vec{F}$ may be written as $\vec{F} = m\dfrac{d\vec{v}}{dt}$

Therefore (1) implies, $\vec{r} \times m\dfrac{d\vec{v}}{dt} = 0$

$$\Rightarrow m\left(\vec{r} \times \frac{d\vec{v}}{dt}\right) = 0$$

$$\Rightarrow \vec{r} \times \frac{d\vec{v}}{dt} = 0 \longrightarrow (2)$$

Now $\dfrac{d}{dt}\left(\vec{r} \times \vec{v}\right)$

$$= \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= \vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= 0 + \vec{r} \times \frac{d\vec{v}}{dt}$$

Thus, $\vec{r} \times \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(\vec{r} \times \vec{v}\right)$

Substituting in (2),

$$\vec{r} \times \frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = 0$$

$$\Rightarrow \vec{r} \times \vec{v} = \vec{h} \text{ (a constant vector in time)} \longrightarrow (3)$$

Multiplying by the mass of the particle m,

$$\Rightarrow m(\vec{r} \times \vec{v}) = m\vec{h} \quad (\text{a constant})$$

$$\Rightarrow \vec{L} = \text{constant}$$

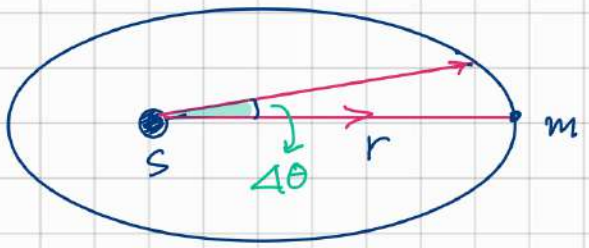
Thus, angular momentum $\vec{L}$ is constant in time.

proved.

Property 3 :

Law of areas : The areal velocity of the particle is constant. In other words, the time rate of change of area is constant.

Proof :



Let us suppose a particle of mass 'm' moves in a central force field generated by the source mass S.

The mass moves a distance of $v\Delta t$ in time Δt.

The area swept by the mass m is

$$\Delta\vec{A} = \frac{1}{2}(\vec{r} \times \vec{v}\Delta t)$$

$$\Rightarrow \frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \vec{v}\right)$$

$$\Rightarrow \frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2m}\left[m\left(\vec{r} \times \vec{v}\right)\right]$$

$$\Rightarrow \frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2m}\vec{L}$$

In the limit $\Delta t \to 0$,

i.e. $$\lim_{\Delta t \to 0} \frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2m}\vec{L}$$

$$\Rightarrow \frac{d\vec{A}}{dt} = \frac{1}{2m}\vec{L} \text{ (a constant).}$$

Thus, the time rate of change of area is constant.

proved.

Property 4 : A central force is conservative in nature.

Mathematically, the expression for central force is -

$$\vec{F} = f(r)\hat{r}$$

We can take the curl of $\vec{F}$ in spherical polar coordinate system.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta} \\ f(r) & 0 & 0 \end{vmatrix} = 0$$

Using Stoke's Theorem we may write

$$\oint \vec{F}.\vec{dl} = \int_s (\vec{\nabla}\times\vec{F}).\vec{ds}$$

$$\Rightarrow \oint \vec{F}.\vec{dl} = 0 \qquad \left[\text{since } \vec{\nabla}\times\vec{F} = 0\right]$$

We know that $\vec{F}.\vec{dl}$ is the small amount of work done by the force $\vec{F}$.

So, $\oint \vec{F}.\vec{dl} = 0$ implies that the work done by the force $\vec{F}$ over a closed loop is 0.

Hence, $\vec{F}$ must be a conservative force.

proved.