* Linearity and Superposition Principle

In physics, we encounter differential equations whose solution describes a physical system we are interested to study.

Linearity of a homogenous differential equation means that if two functions $f(x)$ and $g(x)$ are solutions of the equations, then any linear combination $af(x) + bg(x)$ is a solution of the equation as well.

For instance, consider the differential equation of a simple harmonic oscillator -

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \longrightarrow (1)$$

We obtain two solutions that satisfy the above equation -

$$x_1 = e^{i\omega t} \quad ; \quad x_2 = e^{-i\omega t}$$

We can verify that any linear combination of the two solutions $x_1(t)$ & $x_2(t)$ shall also satisfy the equation.

i.e. $A x_1(t) + B x_2(t)$ will also satisfy the equation, given A and B are constants.

This leads us to the superposition principle :-

"When two or more harmonic oscillations are superimposed, then resultant displacement of a particle undergoing both harmonic oscillations is equal to the vector sum of the displacements of that particle due to individual oscillations at that instant."

You already know the reason behind this -

↳ the equation describing simple harmonic oscillation is a linear homogenous equation.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \longrightarrow (2)$$

The superposition principle also holds for two or more waves propagating simultaneously in an elastic medium since the equation describing the wave motion is a linear homogenous equation.

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \longrightarrow (3)$$

We shall however confine ourself to the study of superposing harmonic oscillators only.

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* Superposition of two collinear harmonic oscillators having equal frequencies.

Consider a system governed by the SHM equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \longrightarrow (4)$$

Let's say we have two solutions i.e. two simple harmonic oscillations of equal frequency but different amplitude and phase acting on a particle whose motion is confined to x-direction only.

$$x_1 = a \sin(\omega t + \phi_1) \longrightarrow (5)$$

$$x_2 = b \sin(\omega t + \phi_2) \longrightarrow (6)$$

We can use the superposition principle to get the resultant –

$$x = x_1 + x_2$$

$$= a \sin(\omega t + \phi_1) + b \sin(\omega t + \phi_2)$$

$$= a \sin(\omega t) \cos(\phi_1) + a \sin(\phi_1) \cos(\omega t)$$

$$+ b \sin(\omega t) \cos(\phi_2) + b \sin(\phi_2) \cos(\omega t)$$

$$= (a \cos\phi_1 + b \cos\phi_2) \sin(\omega t)$$

$$+ (a \sin\phi_1 + b \sin\phi_2) \cos(\omega t)$$

Let us put

$$A\cos\theta = a\cos\phi_1 + b\cos\phi_2 \longrightarrow (7)$$

and $$A\sin\theta = a\sin\phi_1 + b\sin\phi_2 \longrightarrow (8)$$

So, $$x = A\cos\theta\sin(\omega t) + A\sin\theta\cos(\omega t)$$

$$x = A\sin(\omega t + \theta)$$

We see that the resultant oscillation is also a simple harmonic oscillation with amplitude A and phase difference θ.

We must now express A and θ in terms of our original quantities - a, b, ϕ_1 and ϕ_2

Finding A is easy - $(7)^2 + (8)^2$

$$A^2\cos^2\theta + A^2\sin^2\theta = (a\cos\phi_1 + b\cos\phi_2)^2 + (a\sin\phi_1 + b\sin\phi_2)^2$$

$$\Rightarrow A^2(\cos^2\theta + \sin^2\theta) = a^2\cos^2\phi_1 + b^2\cos^2\phi_2 + 2ab\cos\phi_1\cos\phi_2 + a^2\sin^2\phi_1 + b^2\sin^2\phi_2 + 2ab\sin\phi_1\sin\phi_2$$

$$\Rightarrow A^2 = a^2\left(\cos^2\phi_1 + \sin^2\phi_1\right) + b^2\left(\cos^2\phi_2 + \sin^2\phi_2\right) + 2ab\left(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2\right)$$

$$\Rightarrow A^2 = a^2 + b^2 + 2ab\cos(\phi_1 - \phi_2)$$

$$\Rightarrow A = \sqrt{a^2 + b^2 + 2ab\cos(\phi_1 - \phi_2)}$$

↳ This is the amplitude of the resultant oscillation.

Finding the phase angle θ is also easy -

$$(8)/(7) \Rightarrow \frac{A\sin\theta}{A\cos\theta} = \frac{a\sin\phi_1 + b\sin\phi_2}{a\cos\phi_1 + b\cos\phi_2}$$

$$\Rightarrow \tan\theta = \frac{a\sin\phi_1 + b\sin\phi_2}{a\cos\phi_1 + b\cos\phi_2}$$

Question : When is the resultant amplitude A maximum ?

$$A = \sqrt{a^2 + b^2 + 2ab\cos(\phi_1 - \phi_2)}$$

↳ A is maximum when $\cos(\phi_1 - \phi_2)$ is maximum

$$\cos(\phi_1 - \phi_2) = +1$$

$$\Rightarrow \phi_1 - \phi_2 = 2n\pi \quad [n = 1, 2, 3, \ldots]$$

i.e. phase difference between the original oscillations is an even multiple of π

In this case, resultant amplitude is

$$A = \sqrt{a^2 + b^2 + 2ab(1)}$$

$$= \sqrt{(a+b)^2}$$

$$= a + b$$

↳ This is known as constructive interference of the two collinear harmonic oscillators.

Question : When is the resultant amplitude A minimum ?

$$A = \sqrt{a^2 + b^2 + 2ab\cos(\phi_1 - \phi_2)}$$

↳ A is minimum when $\cos(\phi_1 - \phi_2)$ is minimum

$$\cos(\phi_1 - \phi_2) = -1$$

$$\Rightarrow \phi_1 - \phi_2 = (2n+1)\pi \quad [n = 0, 1, 2, \ldots]$$

i.e. phase difference between the original oscillations is an even multiple of π

$$\therefore A = \sqrt{a^2 + b^2 - 2ab}$$

$$= \sqrt{(a-b)^2}$$

$$= a - b$$

↳ This is known as destructive interference of the two collinear harmonic oscillators.

Question : If two harmonic oscillators of equal amplitudes interfere, when is the amplitude maximum and minimum ?

$$A = \sqrt{a^2 + b^2 + 2ab\cos(\phi_1 - \phi_2)}$$

$$\therefore A^2 = a^2 + a^2 + 2a^2\cos(\phi_1 - \phi_2)$$

$$A^2 = 2a^2\left[1 + \cos(\phi_1 - \phi_2)\right]$$

$$A^2 = 2a^2\left[\cos^2\left(\frac{\phi_1 - \phi_2}{2}\right) + \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) + \cos\left(2\frac{\phi_1 - \phi_2}{2}\right)\right]$$

$$A^2 = 2a^2\left[\cos^2\left(\frac{\phi_1 - \phi_2}{2}\right) + \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) + {} + \cos^2\left(\frac{\phi_1 - \phi_2}{2}\right) - \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right)\right]$$

$$= 2a^2\left[2\cos^2\left(\frac{\phi_1 - \phi_2}{2}\right)\right]$$

$$= 4a^2\cos^2\left(\frac{\phi_1 - \phi_2}{2}\right)$$

$$\Rightarrow A = 2a\cos\left(\frac{\phi_1 - \phi_2}{2}\right)$$

The amplitude A is maximum i.e. 2a when.

$$\cos\left(\frac{\phi_1-\phi_2}{2}\right) = \pm 1$$

$$\Rightarrow \frac{\phi_1-\phi_2}{2} = n\pi \quad [n = 0, 1, 2, \ldots]$$

$$\Rightarrow \phi_1 - \phi_2 = 2n\pi$$

i.e. amplitude is maximum when the harmonic oscillators are in phase.

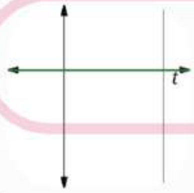
The amplitude A is minimum ie. 0 when

$$\cos\left(\frac{\phi_1-\phi_2}{2}\right) = 0$$

$$\Rightarrow \frac{\phi_1-\phi_2}{2} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \phi_1 - \phi_2 = (2n+1)\pi$$

i.e. amplitude is minimum when the harmonic oscillators are in opposite phase.



Superposition of two harmonic oscillators

Superposition of two harmonic oscillators

www.desmos.com

https://www.desmos.com/calculator/qhahmtpfvr

Exercise :- Verify everything you have learnt above graphically in the desmos graph given in the link.

* Superposition of two collinear harmonic oscillators having unequal frequencies.

[Beat Phenomenon]

Beat Frequency

Here two tuning forks are used to demonstrate beat freque...

youtu.be

https://youtu.be/V8W4Djz6jnY

↳ Demonstration Video for Beat Phenomenon.

When two waves of nearly same frequency and having a constant initial phase difference propagate simultaneously along the same straight line and in the same direction, they superimpose and the interference pattern has a periodic increase and decrease in the amplitude. with a frequency equal to the difference of the two original frequencies.

This phenomenon is called 'beat'

Consider a system governed by the SHM equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \longrightarrow (4)$$

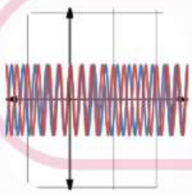
Let's say we have two solutions i.e. two simple harmonic oscillations of unequal frequency but same amplitude and constant phase on a particle whose motion is confined to x-direction only.

$$x_1 = a \sin(\omega_1 t) \longrightarrow (5)$$

$$x_2 = a \sin(\omega_2 t + \phi) \longrightarrow (6)$$

We can use the superposition principle to get the resultant -

$$
\begin{aligned}
x &= x_1 + x_2 \\
&= a \sin(\omega_1 t) + a \sin(\omega_2 t + \phi) \\
&= 2a \sin\left(\frac{\omega_1 t + \omega_2 t + \phi}{2}\right) \cos\left(\frac{\omega_1 t - \omega_2 t - \phi}{2}\right) \\
&= 2a \sin\left(\left[\frac{\omega_1 + \omega_2}{2}\right] t + \frac{\phi}{2}\right) \cos\left(\left[\frac{\omega_1 - \omega_2}{2}\right] t - \frac{\phi}{2}\right)
\end{aligned}
$$



Beat Phenomenon

Beat Phenomenon

www.desmos.com

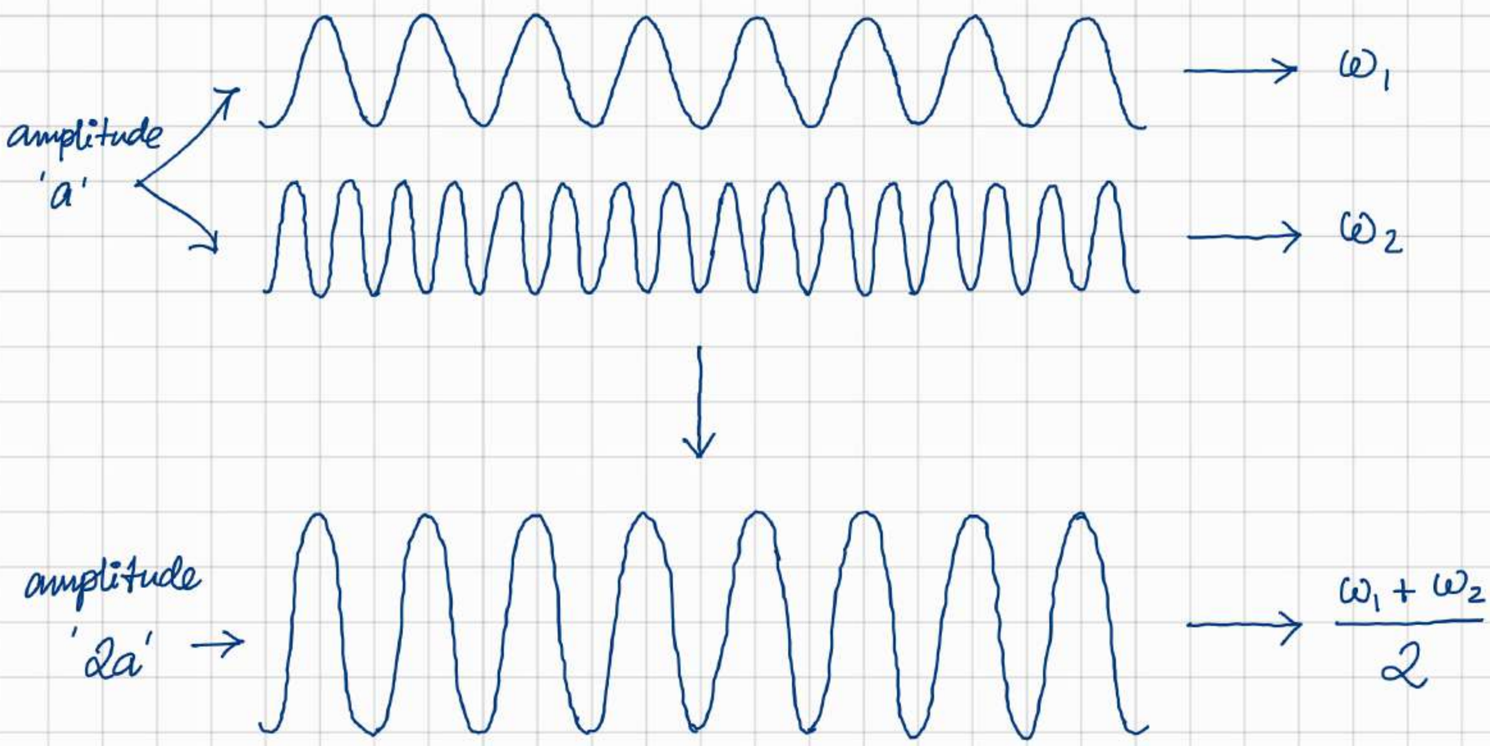
https://www.desmos.com/calculator/njq0kud8iy

↳ Try to understand the above expression using the desmos graph

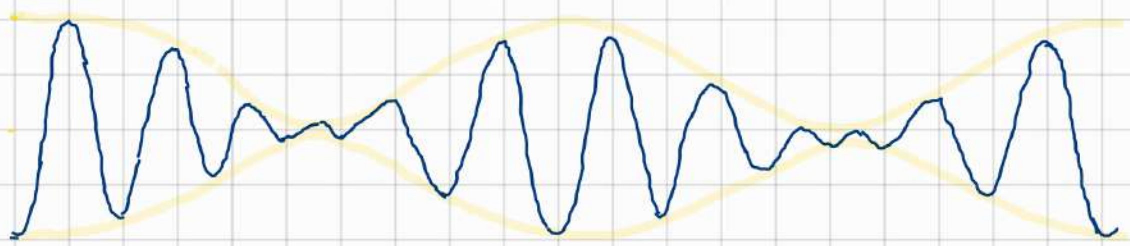
Let's break down the expression into two parts -

$$x = 2a\sin\left(\left[\frac{\omega_1+\omega_2}{2}\right]t + \frac{\phi}{2}\right)\cos\left(\left[\frac{\omega_1-\omega_2}{2}\right]t - \frac{\phi}{2}\right)$$

$2a\sin\left(\left[\frac{\omega_1+\omega_2}{2}\right]t + \frac{\phi}{2}\right)$ is a normal sine wave that has an amplitude $2a$ modulated by the sine wave with a frequency that is an average between the original frequencies ω_1 and ω_2 i.e. $\left[\frac{\omega_1+\omega_2}{2}\right]$



On the other hand, the $\cos\left(\left[\frac{\omega_1-\omega_2}{2}\right]t - \frac{\phi}{2}\right)$ modulates this wave with frequency $\left[\frac{\omega_1-\omega_2}{2}\right]$ giving rise to periodic rise and fall in the amplitude.



Superposition of N collinear Harmonic Oscillator having equal phase differences.

Consider a superposition of N harmonic oscillators each of amplitude a, angular frequency ω and differing in initial phase from its neighbouring oscillation by an angle ϕ.

$$x_1 = a\cos(\omega t)$$
$$x_2 = a\cos(\omega t + \phi)$$
$$x_3 = a\cos(\omega t + 2\phi)$$
$$\vdots$$
$$x_N = a\cos(\omega t + (N-1)\phi)$$

From the superposition principle, the resultant motion is given by:

$$x = a\cos(\omega t) + a\cos(\omega t + \phi) + a\cos(\omega t + 2\phi) + \dots + a\cos(\omega t + (N-1)\phi)$$

Since, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$$x = \mathrm{Re}\left[a\left\{e^{i\omega t} + e^{i(\omega t+\phi)} + e^{i(\omega t + 2\phi)} + \dots e^{i(\omega t + (N-1)\phi)}\right\}\right]$$

$$= \mathrm{Re}\left[ae^{i\omega t}\left\{1 + e^{i\phi} + e^{i2\phi} + \dots + e^{i(N-1)\phi}\right]$$

This is an GP series ; common ratio , $r = e^{i\phi}$

$$S = \frac{a(r^N - 1)}{r - 1}$$

$$= \frac{1\left(e^{iN\phi} - 1\right)}{e^{i\phi} - 1}$$

$$= \frac{e^{iN\frac{\phi}{2}}}{e^{i\frac{\phi}{2}}} \left(\frac{e^{iN\frac{\phi}{2}} - e^{-iN\frac{\phi}{2}}}{e^{i\frac{\phi}{2}} - e^{-i\frac{\phi}{2}}} \right)$$

$$= e^{i(N-1)\frac{\phi}{2}} \left(\frac{\sin\left(N\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} \right)$$

$$\therefore \quad x = \mathrm{Re}\left[a e^{i\omega t} \, e^{i(N-1)\frac{\phi}{2}} \left(\frac{\sin\left(N\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} \right) \right]$$

$$= \mathrm{Re}\left[a \frac{\sin\left(N\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} \, e^{i\left(\omega t + \frac{1}{2}(N-1)\phi\right)} \right]$$

$$= \underbrace{a \frac{\sin\left(N\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}}}_{A} \; \underbrace{\cos\left(\omega t + \frac{1}{2}(N-1)\phi\right)}_{\cos(\omega t + \delta)}$$

$$= A \quad \cos\left(\omega t + \delta\right)$$

Thus, amplitude $A = a \dfrac{\sin\left(N\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}}$

and the resultant phase $\delta = \frac{1}{2}(N-1)\phi$