Superposition of two perpendicular harmonic oscillations

i) Vibration having equal frequencies :-

Suppose that a particle moves under the simultaneous influence of two simple harmonic oscillations of equal frequency, one along X-axis, the other along the perpendicular Y-axis.

The question we want to answer is -
"What is the motion of the particle ?"

The equations of motion governing the systems are -

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

Let's say the solutions of these equations are -

$$x = a \cos(\omega t) \longrightarrow (1)$$

$$y = b \cos(\omega t + \phi) \longrightarrow (2)$$

a and b are the amplitudes of x & y oscillations.

ϕ is the phase difference between them.

$$(1) \Rightarrow \frac{x}{a} = \cos(\omega t) \quad ; \quad \sin^2(\omega t) = 1 - \cos^2(\omega t)$$

$$\Rightarrow \sin(\omega t) = \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

$$(2) \Rightarrow \frac{y}{b} = \cos(\omega t + \delta)$$

$$\Rightarrow \frac{y}{b} = \cos(\omega t)\cos(\delta) - \sin(\omega t)\sin(\delta)$$

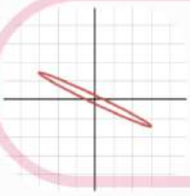
$$\Rightarrow \frac{y}{b} = \frac{x}{a}\cos(\delta) - \sqrt{\left(1 - \frac{x^2}{a^2}\right)}\sin(\delta)$$

$$\Rightarrow \left(\frac{x}{a}\cos(\delta) - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2(\delta)$$

$$\Rightarrow \frac{x^2}{a^2}\cos^2(\delta) + \frac{y^2}{b^2} - 2\frac{x}{a}\cdot\frac{y}{b}\cos(\delta) = \sin^2(\delta) - \frac{x^2}{a^2}\sin^2(\delta)$$

$$\Rightarrow \frac{x^2}{a^2}\left(\cos^2(\delta) + \sin^2(\delta)\right) + \frac{y^2}{b^2} - 2\frac{x}{a}\frac{y}{b}\cos(\delta) = \sin^2(\delta)$$

$$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{x}{a}\frac{y}{b}\cos(\delta) = \sin^2(\delta)} \rightarrow (3)$$



Lissajous Figure

Lissajous Figure

www.desmos.com

https://www.desmos.com/calculator/jbljtg7gqn

Equation (3) is the general equation of an ellipse whose axes are inclined to the coordinate axes.

Verify this is the desmos graph provided in the link.

Hence, the path followed by the particle, which is subjected to the two rectangular SHMs of equal frequencies, is, in general an ellipse.

Special Cases :

Case I : if there is no phase difference i.e. $\delta = 0$

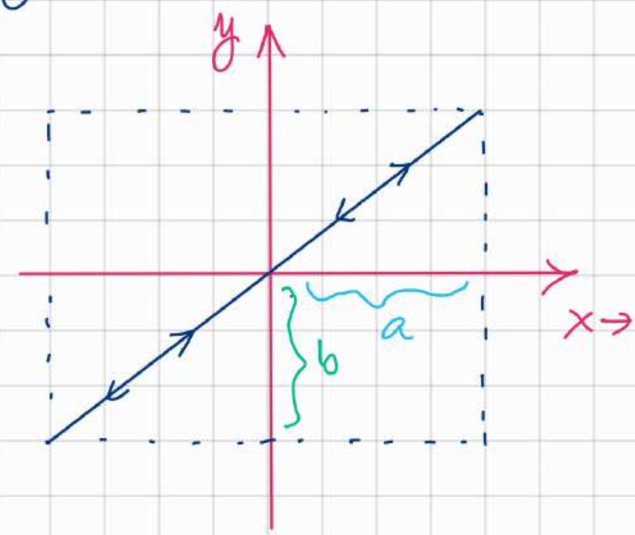
then equation (3) reduces to :-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$\Rightarrow \frac{x}{a} - \frac{y}{b} = 0$$

$$\Rightarrow y = \frac{b}{a} x$$



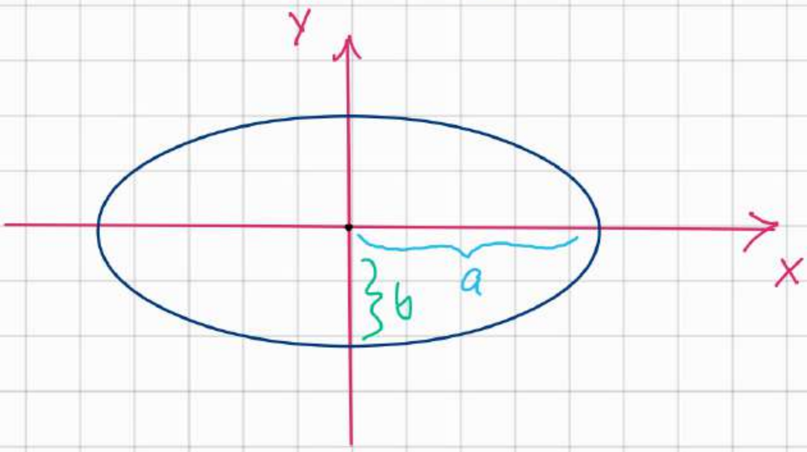
This gives a pair of coincident straight lines having positive slope $\frac{b}{a}$ and passing through the origin.

Case II : $\delta = \frac{\pi}{2}$

then equation 3 reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

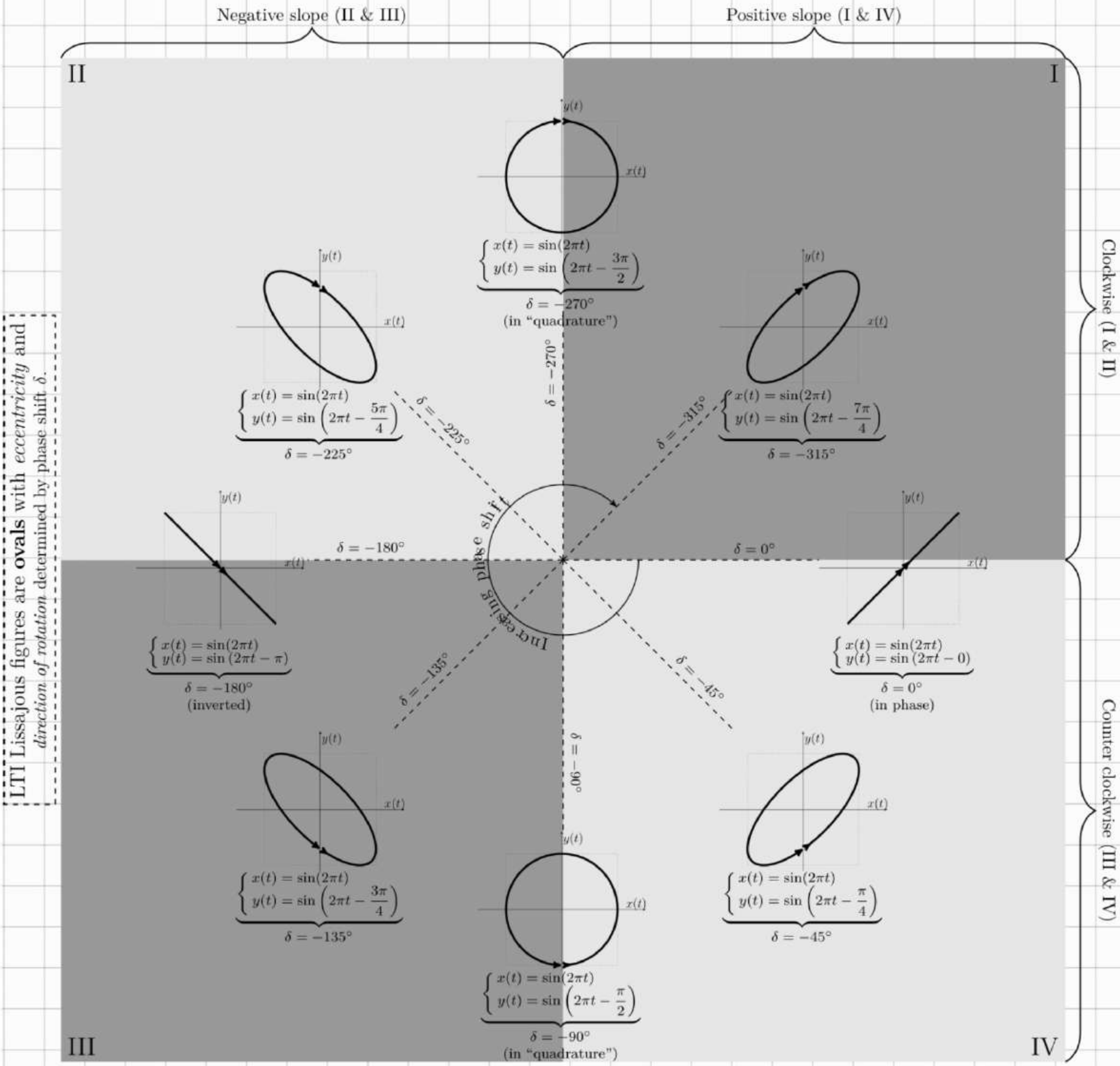
which is the equation of an ellipse whose principal axis lie along the X and Y axes.



Question : What is a Lissajous figure ?

↳ Lissajous figure is a curve formed by combining two simple harmonic oscillators that are perpendicular to each other.

Uses : can be used to determine the frequency, amplitude and phase relationships of harmonic variables.


Negative slope (II & III)
Positive slope (I & IV)
II
I
III
IV
Clockwise (I & II)
Counter clockwise (III & IV)
LTI Lissajous figures are **ovals** with *eccentricity* and *direction of rotation* determined by phase shift δ.
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin\left(2\pi t - \frac{3\pi}{2}\right) \end{cases}$
$\delta = -270°$
(in "quadrature")
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin\left(2\pi t - \frac{5\pi}{4}\right) \end{cases}$
$\delta = -225°$
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin\left(2\pi t - \frac{7\pi}{4}\right) \end{cases}$
$\delta = -315°$
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin(2\pi t - \pi) \end{cases}$
$\delta = -180°$
(inverted)
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin(2\pi t - 0) \end{cases}$
$\delta = 0°$
(in phase)
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin\left(2\pi t - \frac{3\pi}{4}\right) \end{cases}$
$\delta = -135°$
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin\left(2\pi t - \frac{\pi}{4}\right) \end{cases}$
$\delta = -45°$
$\begin{cases} x(t) = \sin(2\pi t) \\ y(t) = \sin\left(2\pi t - \frac{\pi}{2}\right) \end{cases}$
$\delta = -90°$
(in "quadrature")
Increasing phase shift
$\delta = -270°$
$\delta = -225°$
$\delta = -315°$
$\delta = -180°$
$\delta = 0°$
$\delta = -135°$
$\delta = -45°$
$\delta = -90°$
$x(t)$
$y(t)$