Velocity of Transverse Waves in a Stretched String.

The velocity is governed by three laws :-

1. The fundamental frequency is inversely proportional to the length of the string.

$$f \propto \frac{1}{l}$$

2. The fundamental frequency is directly proportional to the square root of the stretching force or tension.

$$f \propto \sqrt{T}$$

3. The fundamental frequency is inversely proportional to the square root of the mass per unit length of the string.

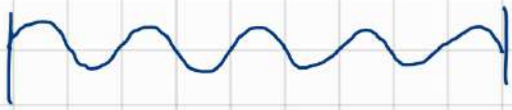
$$f \propto \frac{1}{\sqrt{m}}$$

Combining, we can write

$$f \propto \frac{1}{l}\sqrt{\frac{T}{m}} \quad \text{or} \quad f = k\,\frac{1}{l}\sqrt{\frac{T}{m}}$$

It has been determined that $k = \frac{1}{2}$.

$$\therefore \quad f = \frac{1}{2l}\sqrt{\frac{T}{m}}$$



If the string vibrates in 'p' number of segments then length of each segment is $\frac{l}{p}$

a length of l/p corresponds to $\frac{\lambda}{2}$.

$$\frac{l}{p} = \frac{\lambda}{2} \Rightarrow \lambda = \frac{2l}{p}$$

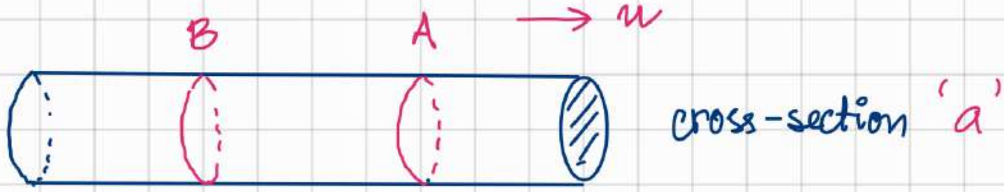
velocity, $v = f\lambda$

$$= \frac{1}{2l}\sqrt{\frac{T}{m}} \times \frac{2l}{p}$$

$$= \frac{1}{p}\sqrt{\frac{T}{m}}$$

Velocity of sound in a hollow tube filled with gas.

Consider a hollow tube filled with gas which has a cross-section of a.



Suppose sound waves travel from left to right with a velocity 'u' in a region of normal density.

Suppose A is a cross section of normal density
B is a cross section of compression

At A		At B	
pressure	P	pressure	P_1
density	ρ	density	ρ_1
velocity	u	velocity	u_1

As there is no change in the average density of the medium, the masses of the medium crossing the sections A and B in one second are equal.

mass of medium entering A in one second

$$a \times u \times \rho$$

mass of medium leaving B in one second.

$$a \times u_1 \times \rho_1$$

$$m = au\rho = au_1\rho_1 \Rightarrow \frac{u_1}{u} = \frac{\rho}{\rho_1}$$

If B is at a region of compression, its density will be higher than the normal density ρ at A.

$$\rho_1 > \rho$$

Hence, $u_1 > u$

The momentum per second of the medium entering A $= mu$

The momentum per second of the medium leaving B $= mu_1$

change in momentum per second

$$= mu - mu_1$$

change in momentum per second is due to the change in pressure $(P_1 - P)$ between A and B.

$$mu - mu_1 = (P_1 - P)\, a$$

$$\Rightarrow mu\left(1 - \frac{u_1}{u}\right) = (P_1 - P)\, a$$

$$\Rightarrow au\rho \cdot u\left(1 - \frac{u_1}{u}\right) = (P_1 - P)\, a$$

$$\Rightarrow u^2\rho\left(1 - \frac{\rho}{\rho_1}\right) = P_1 - P$$

$$\Rightarrow u^2\rho = \frac{P_1 - P}{\left(\frac{\rho_1 - \rho}{\rho_1}\right)}$$

We know, Bulk modulus of elasticity

$$K = \frac{(P_1 - P)}{\left(\frac{V - V_1}{V}\right)}$$

Since, $V \propto \frac{1}{\rho}$

$$\frac{V_1 - V}{V} = \frac{\rho_1 - \rho}{\rho_1}$$

$$\Rightarrow K = \frac{P_1 - P}{\left(\frac{\rho_1 - \rho}{\rho_1}\right)}$$

$$\therefore \quad u^2 \rho = K$$

$$\Rightarrow u = \sqrt{\frac{K}{\rho}}$$

This is applicable for plane waves of small amplitude i.e. sound waves of ordinary intensity.

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Newton's Formula for velocity of sound in air

Newton assumed that temperature remains constant when sound waves travel through air

i.e. the process is isothermal.

Sound waves travel through air as longitudinal waves constituting of compressions and rarefaction

Suppose initial pressure → P
initial volume → V

increase in pressure → p
decrease in volume → v

final pressure → $P+p$
final volume → $V-v$

Applying Boyle's law $P \propto \frac{1}{V}$.

$$(P+p)(V-v) = PV$$

$$\Rightarrow PV + pV - Pv - pv = PV$$

pv is so small that it can be ignored

$$\therefore P = \frac{pV}{v}$$

We know Bulk modulus,

$$K = \frac{\text{change in pressure}}{\left(\frac{\text{change in volume}}{\text{original volume}}\right)}$$

$$\Rightarrow K = \frac{p}{v/V} = \frac{pV}{v} = P$$

Thus, velocity of sound in air is

$$u = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{P}{\rho}}$$

$$\boxed{u = \sqrt{\frac{P}{\rho}}}$$

Laplace's Correction

Laplace reasoned that

during compression particles come near each other and gets heated up

during rarefaction particles go apart and temperature goes down.

Hence, the process is not isothermal but rather adiabatic.

For an adiabatic process,

$$PV^\gamma = \text{const.}$$

initial pressure = P
initial volume = V

change in pressure = p
change in volume = v

$$\therefore \quad PV^\gamma = (P+p)(V-v)^\gamma$$

$$\Rightarrow PV^\gamma = P\left(1+\frac{p}{P}\right) V^\gamma \left(1-\frac{v}{V}\right)^\gamma$$

$$\Rightarrow 1 = \left(1+\frac{p}{P}\right)\left(1-\frac{v}{V}\right)^\gamma$$

Using binomial expansion we can write.

$$\left(1-\frac{v}{V}\right)^{\gamma} \text{ as } 1-\frac{\gamma v}{V}+\ldots$$

Hence,
$$1=\left(1+\frac{p}{P}\right)\left(1-\frac{\gamma v}{V}\right)$$

$$\Rightarrow 1=1-\frac{\gamma v}{V}+\frac{p}{P}-\frac{\gamma p v}{PV}$$

Ignoring $\frac{\gamma p v}{PV}$,

$$\frac{p}{P}=\frac{\gamma v}{V}$$

$$\Rightarrow \gamma P=\frac{pV}{v}$$

$$\Rightarrow \gamma P=\frac{p}{v/V}$$

$$\Rightarrow \gamma P=K \quad \text{(Bulk modulus)}$$

$$\therefore \quad \text{velocity } u=\sqrt{\frac{K}{\rho}}=\sqrt{\frac{\gamma P}{\rho}}$$

Considering 1 atm pressure, and 0.001293 g/cc density of air,

Newton

$$u = \sqrt{\frac{P}{\rho}}$$

$= 280$ m/s

Laplace.

$\gamma = 1.42$

$$u = \sqrt{\frac{\gamma P}{\rho}}$$

$= 332.8$ m/s.

Experimental value is 330 m/s.