## Why gamma function ?

The gamma function is probably the special function that occurs most frequently in the discussion of problems in Physics.

Although the gamma function does not usually describe a physical quantity of interest but it appears as a factor of expansions of physically relevant quantities. This is because most physical theories involve quantities governed by differential equations and it has been shown that the gamma function is one of a general class of functions that do not satisfy any differential equation with rational coefficients.

We shall study three definitions of Gamma function :

## DEFINITION 1 : Infinite Limit (by Euler)

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1.2.3.\cdots n}{z(z+1)(z+2)\cdots(z+n)}\, n^{z}$$

$z \neq 0, -1, -2, -3, \ldots$ ($z$ may be complex)

The important question to ask is what consequence would this definition have that might aid in our calculations encountered in physical problems ?

We shall obtain a very useful functional relation for the gamma function. Let's see what it is -

Replace $z$ by $z+1$ $\longrightarrow$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1.2.3\ldots n}{(z+1)(z+2)\cdots(z+n+1)} n^{z+1}$$

$$= \lim_{n\to\infty} \left(\frac{nz}{z+n+1}\right) \cdot \frac{1.2.3\ldots n}{z(z+1)(z+2)\ldots(z+n)} n^{z}$$

Calculating the limit, $\lim_{n\to\infty} \left(\frac{nz}{z+n+1}\right)$

$$\lim_{n\to\infty} \frac{\frac{d}{dn}(nz)}{\frac{d}{dn}(z+n+1)} = \frac{z}{1} = z.$$

$$\therefore \quad \boxed{\Gamma(z+1) = z\,\Gamma(z)}$$ $\longrightarrow$ Remember this !

For eg: $\Gamma(4) = 3\,\Gamma(3)$

$$= 3 \cdot 2 \cdot \Gamma(1)$$

Wait ! What is $\Gamma(1)$ ?

Let's get back to the definition -

$$\Gamma(z) = \lim_{n\to\infty} \frac{1.2.3\ldots n}{z(z+1)(z+2)\ldots(z+n)} n^{z}$$

Putting $z = 1$,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1.2.3\ldots n}{1.2.3\ldots n.(n+1)} n^1$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

Thus, $\boxed{\Gamma(1) = 1}$ → Remember this!

$$\therefore \Gamma(4) = 3\,\Gamma(3)$$
$$= 3.2\,\Gamma(1)$$
$$= 3.2.1$$
$$= 3!$$

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$= n(n-1)\,\Gamma(n-1)$$
$$= n(n-1)(n-2)\ldots\Gamma(1)$$
$$= n(n-1)(n-2)\ldots 1$$
$$= n!$$

→ Remember this!

Summarising what we have learnt from this definition:

1. $\Gamma(n+1) = n\Gamma(n)$
2. $\Gamma(1) = 1$
3. $\Gamma(n+1) = n!\ /\ \Gamma(n) = (n-1)!$

## DEFINITION 2 : Definite Integral (by Euler)

$$\Gamma(z) \equiv \int_0^{\infty} e^{-t} t^{z-1} dt \quad ; \quad Re(z) > 0$$

Why do we put the restriction $Re(z) > 0$?

↳ We do it in order to avoid divergence of the integral. The proof is beyond our discussion.

This is an integral which you will encounter again and again in physical problems. One important skill for a physicist to have is being able to convert certain integrals to the gamma integral form and carry out the calculations.

Let's do some problems -

**Problem 1 :** Calculate the average energy per molecule for a structureless classical ideal gas at room temperature $T$.

**Soln :** According to Maxwell-Boltzmann statistics, a state of energy $E$ is occupied with a probability

$$p(E) = Ce^{-\beta E} \quad \text{where } \beta = \frac{1}{kT}, \; k \rightarrow \text{Boltzmann constant.}$$

Let's say, the number of states in a small energy interval $dE$ at $E$ is $n(E)\, dE$.

For a structureless ideal gas, $n(E) \propto E^{\frac{1}{2}}$ where $E$ is the kinetic energy of a gas molecule. and $E \in (0,\infty)$

total probability of the states at energy $E$ is thus,

$$C\, n(E)\, e^{-\beta E}\, dE = C\, E^{\frac{1}{2}} e^{-\beta E}\, dE$$

where $C$ is the normalisation constant.

Since total probability in the range $(0,\infty)$ must be 1,

$$\therefore \; C \int_0^{\infty} E^{\frac{1}{2}} e^{-\beta E}\, dE = 1$$

Now can we cast the LHS in the below form?

$$\Gamma(z) \equiv \int_0^{\infty} e^{-t}\, t^{z-1}\, dt$$

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$$\frac{C}{\beta^{3/2-1}\,\beta^{1}} \int_0^{\infty} e^{-\beta E} (\beta E)^{3/2-1}\, d(\beta E) = 1$$

$$\Rightarrow \frac{C}{\beta^{3/2}} \int_0^{\infty} e^{-\beta E} (\beta E)^{3/2-1}\, d(\beta E) = 1$$

$$\Rightarrow \frac{C}{\beta^{3/2}}\, \Gamma(3/2) = 1$$

$$\Rightarrow C = \frac{\beta^{3/2}}{\Gamma(3/2)}$$

But what is the value of $\Gamma(3/2)$ ?

There is nothing that prevents us from writing

$$\Gamma(3/2) = \Gamma\left(\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \quad \text{using } \Gamma(n+1) = n\Gamma(n)$$

So, the we would rather be asking what is the value of $\Gamma\left(\frac{1}{2}\right)$ ?

Well $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt$

$$= \int_0^\infty e^{-t}\, t^{-\frac{1}{2}}\, dt$$

Let $y = t^{1/2}$

$$\Rightarrow dy = \frac{1}{2} t^{-\frac{1}{2}}\, dt$$

$$\Rightarrow t^{-1/2}\, dt = 2\, dy.$$

Also, $t = y^2$

Thus, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2}\, 2\, dy.$

$$= 2\int_0^\infty e^{-y^2}\, dy.$$

Since $e^{-y^2}$ is an even function,

so $2\int_0^\infty e^{-y^2} dy = \int_{-\infty}^{+\infty} e^{-y^2} dy.$

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$$\therefore\ \Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{+\infty} e^{-y^2}\, dy.$$

How would you calculate this !!!

Well, $\int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\int_{-\infty}^{+\infty} e^{-y^2} dy \int_{-\infty}^{+\infty} e^{-y^2} dy}$

$$= \sqrt{\int_{-\infty}^{+\infty} e^{-y^2} dy \int_{-\infty}^{+\infty} e^{-x^2} dx}$$

$$= \sqrt{\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy}$$

Basically we need to integrate $e^{-(x^2+y^2)}$ over the whole x-y space.

Let's switch to polar coordinates –

$$r = \sqrt{x^2+y^2}$$

$\therefore$ $e^{-(x^2+y^2)}$ becomes $e^{-r^2}$

So, now we have to integrate $e^{-r^2}$ over $r$-$\phi$ plane.

Thus, $$\sqrt{\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy} = \sqrt{\int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,dr\,r\,d\phi}$$

$$\int_0^{\infty}\left(re^{-r^2}\int_0^{2\pi} d\phi\right)dr$$

$$= \int_0^{\infty}\left(re^{-r^2}\cdot 2\pi\right)dr$$

Let $r^2 = m$

$\Rightarrow 2r\,dr = dm$

$\Rightarrow r\,dr = \frac{dm}{2}$

$$= 2\pi \int_0^{\infty} re^{-r^2}\,dr$$

$$= 2\pi \int_0^{\infty} e^{-m}\,\frac{dm}{2}$$

$$= \pi \int_0^{\infty} e^{-m}\,dm$$

$$= \pi \left(-e^{-m}\Big|_0^{\infty}\right)$$

$$= \pi \left(e^{-0} - e^{-\infty}\right)$$

$$= \pi$$

Thus, $\sqrt{\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy} = \sqrt{\pi}$

$$\Rightarrow \sqrt{\int_{-\infty}^{\infty} e^{-y^2}\, dy} = \sqrt{\pi}$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Rightarrow \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \Gamma\left(3/2\right) = \frac{\sqrt{\pi}}{2}$$

$$\therefore\ C = \frac{\beta^{3/2}}{\sqrt{\pi}/2} = 2\,\frac{\beta^{3/2}}{\sqrt{\pi}}$$

Thus, the probability distribution of states at kinetic energy E takes the form:

$$p(E) = 2\,\frac{\beta^{3/2}}{\sqrt{\pi}} \cdot E^{1/2} e^{-\beta E}$$

To find out the average kinetic energy $\langle E \rangle$, we need to perform the integration

$$\langle E \rangle = \int_0^\infty E \, p(E) \, dE$$

$$= 2 \frac{\beta^{3/2}}{\sqrt{\pi}} \int_0^\infty E \cdot E^{1/2} e^{-\beta E} \, dE$$

$$= 2 \frac{\beta^{3/2}}{\sqrt{\pi}} \int_0^\infty E^{3/2} e^{-\beta E} \, d(\beta E) \frac{1}{\beta}$$

$$= \frac{2}{\beta\sqrt{\pi}} \int_0^\infty (\beta E)^{3/2} e^{-\beta E} \, d(\beta E)$$

$$= \frac{2}{\beta\sqrt{\pi}} \int_0^\infty (\beta E)^{5/2 - 1} e^{-\beta E} \, d(\beta E)$$

$$= \frac{2}{\beta\sqrt{\pi}} \Gamma(5/2)$$

$$= \frac{2}{\beta\sqrt{\pi}} \Gamma(3/2 + 1)$$

$$= \frac{2}{\beta\sqrt{\pi}} \cdot \frac{3}{2} \Gamma(3/2)$$

$$= \frac{2}{\beta\sqrt{\pi}} \cdot \frac{3}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{3}{2\beta} = \frac{3}{2} kT$$

Problem 2 : When the gamma function appears in physical problems, it is often in some variation. It is important for you to recognize the variations as a gamma function.

Show that, $\Gamma(z)$ may be written as -

(i) $\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt$

(ii) $\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt$

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Soln 2(i).

$$2\int_0^\infty e^{-t^2} t^{2z-1}\, dt$$

Let, $t^2 = m$

$$\Rightarrow 2t\,dt = dm \Rightarrow dt = \frac{dm}{2t}$$

$$\therefore 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt = 2\int_0^\infty e^{-m} t^{2z-1} \frac{dm}{2t}$$

$$= \int_0^\infty e^{-m} t^{2z-2}\, dm$$

$$= \int_0^\infty e^{-m} t^{2(z-1)}\, dm$$

$$= \int_0^\infty e^{-m} m^{z-1}\, dm = \Gamma(z)$$

Sol$^n$ 2 (ii).
$$\int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt$$

Let $e^{-m} = t$

$\Rightarrow -m = \ln t$

$\Rightarrow m = -\ln(t)$ or $m = \ln\left(\frac{1}{t}\right)$

$\Rightarrow dm = -\frac{1}{t}\,dt$

$\Rightarrow dt = -t\,dm$

$= -e^{-m}\,dm$

Since, $e^{-m} = t$

if $t \to 0$, $m \to \infty$

if $t \to 1$, $m \to 0$

$$\therefore \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt = \int_\infty^0 m^{z-1}\left(-e^{-m}\right)dm$$

$$= \int_0^\infty e^{-m} m^{z-1}\,dm$$

$$= \Gamma(z)$$

# A FEW PRACTICE PROBLEMS

**Problem 1 :**

$$\left[ \Gamma(z) = \lim_{n\to\infty} \frac{1.2.3.\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z} \qquad z \neq 0, -1, -2, -3, \ldots \ (z \text{ may be complex}) \right]$$

Well, this definition allowed you to derive the relation $\Gamma(z+1) = z!$

I want you to use the definite integral definition of the gamma function to derive this relation.

$$\left[ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \right]$$

Hint : Start by calculating the integral $\int_0^{\infty} e^{-\alpha x} dx$.

**Solution :**

$$\int_0^{\infty} e^{-\alpha x} dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^{\infty} = \frac{1}{\alpha}$$

Let us differentiate repeatedly w.r.t. to $\alpha$

$$\frac{d}{d\alpha}\int_0^\infty e^{-\alpha x}\,dx = \frac{d}{d\alpha}\left(\frac{1}{\alpha}\right)$$

$$\Rightarrow \int_0^\infty -x e^{-\alpha x}\,dx = -\frac{1}{\alpha^2}$$

$$\Rightarrow \int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

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$$\frac{d}{d\alpha}\int_0^\infty x e^{-\alpha x}\,dx = \frac{d}{d\alpha}\left(\frac{1}{\alpha^2}\right)$$

$$\Rightarrow -\int_0^\infty x^2 e^{-\alpha x}\,dx = -\frac{1.2}{\alpha^3}$$

$$\Rightarrow \int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{1.2}{\alpha^3} = \frac{2!}{\alpha^3}$$

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$$\frac{d}{d\alpha}\int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{d}{d\alpha}\left(\frac{1.2}{\alpha^3}\right)$$

$$\Rightarrow \int_0^\infty x^3 e^{-\alpha x}\,dx = \frac{1.2.3}{\alpha^4} = \frac{3!}{\alpha^4}$$

In general, $$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

Putting $\alpha = 1$ and $n = z-1$

$$\int_0^\infty x^{z-1} e^{-x} dx = (z-1)!$$

We did it !

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Problem 2 : Now prove the relation $\Gamma(z+1) = z\Gamma(z)$ using the definite integral definition of gamma function.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Replacing $z$ by $z+1$

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt$$

Integrating by parts -

$$\Gamma(z+1) = t^z \int e^{-t} dt \Big|_0^\infty - \int_0^\infty \left\{ \left(-e^{-t}\right) \frac{d}{dt}\left(t^z\right) \right\} dt$$

$$= -t^z e^{-t}\Big|_0^{\infty} + \int_0^{\infty} z t^{z-1} e^{-t}\, dt$$

$$= (0-0) + z\int_0^{\infty} e^{-t} t^{z-1}\, dt.$$

$$= z\,\Gamma(z)$$

We did it again!

Problem 3 : Express the following integrals as $\Gamma$ functions and evaluate them whenever possible.

(i) $$\int_0^{\infty} x^{2/3} e^{-x}\, dx = \int_0^{\infty} x^{5/3-1} e^{-x}\, dx$$

$$= \Gamma(5/3)$$

(ii) $$\int_0^{\infty} x^{-1/2} e^{-x}\, dx = \int_0^{\infty} x^{\frac{1}{2}-1} e^{-x}\, dx$$

$$= \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

(iii) $$\int_0^{\infty} \sqrt{x}\, e^{-x}\, dx = \int_0^{\infty} x^{3/2-1} e^{-x}\, dx$$

$$= \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

(iv) $\int_0^{\infty} x^2 e^{-x^2} dx$

Let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$\int_0^{\infty} t\, e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^{\infty} t^{\frac{1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{3/2-1} e^{-t}\, dt$$

$$= \frac{1}{2}\,\Gamma(3/2)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}$$

(v) $\int_0^{\infty} x e^{-x^3} dx$

Let, $x^3 = t$

$\Rightarrow 3x^2 dx = dt$

$\Rightarrow dx = \frac{dt}{3x^2} = \frac{dt}{3t^{2/3}}$

$$\therefore \int_0^\infty x e^{-x^3} dx = \int_0^\infty t^{1/3} e^{-t} \frac{dt}{3t^{2/3}}$$

$$= \frac{1}{3} \int_0^\infty t^{1/3 - 2/3} e^{-t} dt$$

$$= \frac{1}{3} \int_0^\infty t^{-1/3} e^{-t} dt$$

$$= \frac{1}{3} \int_0^\infty t^{2/3 - 1} e^{-t} dt$$

$$= \frac{1}{3} \Gamma\left(2/3\right)$$

(vi) $\displaystyle\int_0^1 x^2 \left(\ln \frac{1}{x}\right)^3 dx$

Let $x = e^{-m}$

$$\Rightarrow \ln x = -m$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = m$$

$$\Rightarrow \frac{1}{1/x}\left(-\frac{1}{x^2}\right) dx = dm$$

$$\Rightarrow -\frac{1}{x} dx = dm$$

$$\Rightarrow dx = -x\, dm$$

$$= -e^{-m} dm$$

$$\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx = \int_\infty^0 e^{-2m} m^3 \left(-e^{-m}\, dm\right)$$

$$= \int_0^\infty e^{-3m} m^3\, dm$$

$$= \frac{1}{3^4}\int_0^\infty e^{-3m} (3m)^3\, d(3m)$$

$$= \frac{1}{81}\Gamma(4)$$

$$= \frac{6}{81} = \frac{2}{27}$$

# The Gamma Function of Negative Numbers.

We can use the recursion relation $\Gamma(z+1) = z\Gamma(z)$ to find the gamma function of negative numbers.

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Rightarrow \Gamma(z) = \frac{1}{z}\Gamma(z+1)$$

For example,

$$\Gamma(-1/2) = \frac{1}{(-1/2)} \cdot \Gamma(-\tfrac{1}{2}+1) = -2\Gamma(1/2) = -2\sqrt{\pi}$$

or $\Gamma(-1.5) = \frac{1}{-1.5}\Gamma(-0.5) = \frac{1}{(-1.5)} \cdot \frac{1}{(-0.5)}\Gamma(0.5)$

Wait! What is $\Gamma(0)$?

$$\Gamma(0) = \frac{1}{0}\Gamma(1) = \frac{1}{0}$$

or we can say that $\Gamma(z) = \frac{\Gamma(z+1)}{z} \to \infty$ as $z \to 0$

So, $\Gamma(0) = \infty$

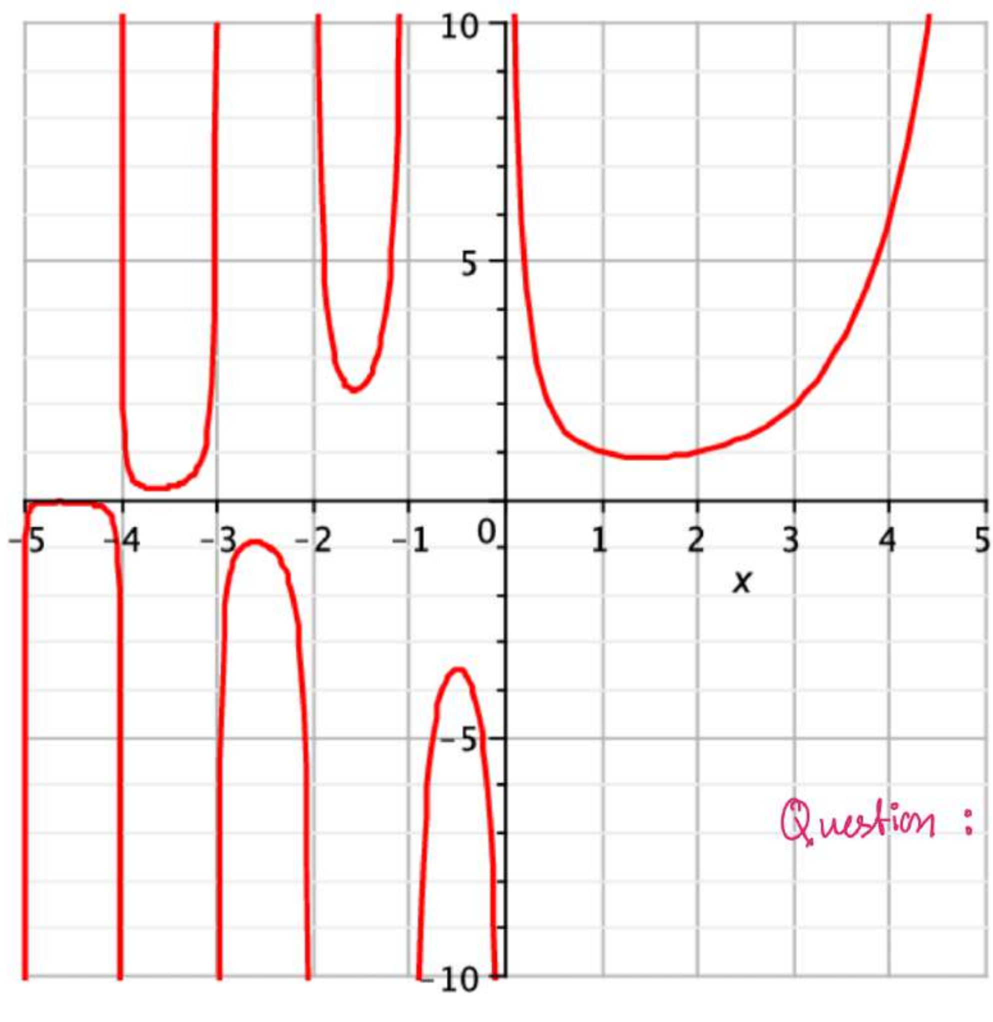
But what is $\Gamma(-1)$ or $\Gamma(-2)$ ?

$$\Gamma(-1) = \frac{1}{-1}\Gamma(0)$$

or $$\Gamma(-2) = \frac{1}{-2}\Gamma(-1) = \frac{1}{-2}\cdot\frac{1}{-1}\Gamma(0)$$

Thus, $\Gamma(z)$ becomes infinite not only at 0 but also at all negative numbers.

Let's take a look at the sketch of the Gamma function.




Question: List all the features that you notice.

1. $\Gamma(z) \to \infty$ as $z \to \infty$

2. $\Gamma(0) \to \infty$

3. $\Gamma(z)$ is discontinuous at all negative integers

4. In the intervals between negative integers, it is alternately positive and negative.

Q : Prove that (i) $\frac{d}{dz}\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \ln t \, dt$

(ii) $\frac{d^n}{dz^n}\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} (\ln t)^n \, dt$

(i) $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$

$\Rightarrow \frac{d}{dt}\Gamma(z) = t^{z-1} e^{-t}$

$\Rightarrow \frac{d}{dt}\left(\frac{d}{dz}\Gamma(z)\right) = \frac{d}{dz}\left(t^{z-1} e^{-t}\right) = t^{z-1} \ln t \, e^{-t}$

↳ since $z$ and $t$ are independent

$\Rightarrow \frac{d}{dz}\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \ln t \, dt$

'n' successive differentiations gives $(\ln t)^n$ term which can prove (ii). Try this.