THE BETA FUNCTION

Let's straight away jump into the definition :-

The beta function is defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

which converges for $m>0$, $n>0$ where m and n are, in general, real numbers.

The beta function may also be written in terms of the gamma function as :

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Let's try to prove it -

$$\Gamma(n)\Gamma(m) = \left[2\int_0^\infty x^{2n-1} e^{-x^2}\,dx\right]\left[2\int_0^\infty y^{2m-1} e^{-y^2}\,dy\right]$$

$$= 4\int_0^\infty\int_0^\infty x^{2n-1}\, y^{2m-1}\, e^{-(x^2+y^2)}\,dx\,dy.$$

Let us change variables to plane polar coordinates.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\Gamma(n)\Gamma(m) = 4\int_0^{\pi/2}\int_0^{\infty} r^{2n-1}\cos^{2n-1}\theta\; r^{2m-1}\sin^{2n-1}\theta\, e^{-r^2}\,dr.\,r d\theta$$

$$= 4\left[\int_0^{\infty} r^{2(m+n-1)} e^{-r^2}\, r\,dr\right]\left[\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta\right]$$

$$= 4\left[\int_0^{\infty} r^{2(m+n)-1} e^{-r^2}\,dr\right]\left[\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta\right]$$

$$= \Gamma(m+n).\,2\int_0^{\pi/2}\left(\sin^{2m-1}\theta\right)\left(\cos^{2n-1}\theta\right)d\theta$$

Let's go back to our definition of the beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let, $x = \sin^2\theta$ $\quad \therefore\ 1-x = 1-\sin^2\theta = \cos^2\theta$

$$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$$

$$\therefore\ B(m,n) = \int_0^{\pi/2}\sin^{2(m-1)}\theta\,\cos^{2(n-1)}.\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2m-1}\cos^{2n-1}\,d\theta$$

Thus, $\Gamma(n)\Gamma(m) = \Gamma(m+n)\, B(m,n)$

$$\Rightarrow B(m,n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}$$

What do we need to remember?

Whenever we see an integral of the form

$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ you must be able to identify it as a beta function $B(m,n)$

and once you can identify the beta function, you can calculate it using the gamma function.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Of course there may be other forms of the beta function such as

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

↳ can you derive this form by substituting $x = (1+y)^{-1}$ in the original definition?

A FEW PRACTICE PROBLEMS

Problem 1 :

$$\left[\Gamma(z) = \lim_{n\to\infty} \frac{1.2.3.\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z} \qquad z \neq 0, -1, -2, -3, \ldots \text{ (z may be complex)} \right]$$

Well, this definition allowed you to derive the relation $\Gamma(z+1) = z!$

I want you to use the definite integral definition of the gamma function to derive this relation.

$$\left[\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \right]$$

Hint : Start by calculating the integral $\int_0^\infty e^{-\alpha x} dx$.

Solution :

$$\int_0^\infty e^{-\alpha x} dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

Let us differentiate repeatedly w.r.t. to α

$$\frac{d}{d\alpha}\int_0^\infty e^{-\alpha x}\,dx = \frac{d}{d\alpha}\left(\frac{1}{\alpha}\right)$$

$$\Rightarrow \int_0^\infty -x e^{-\alpha x}\,dx = -\frac{1}{\alpha^2}$$

$$\Rightarrow \int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

$$\frac{d}{d\alpha}\int_0^\infty x e^{-\alpha x}\,dx = \frac{d}{d\alpha}\left(\frac{1}{\alpha^2}\right)$$

$$\Rightarrow -\int_0^\infty x^2 e^{-\alpha x}\,dx = -\frac{1.2}{\alpha^3}$$

$$\Rightarrow \int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{1.2}{\alpha^3} = \frac{2!}{\alpha^3}$$

$$\frac{d}{d\alpha}\int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{d}{d\alpha}\left(\frac{1.2}{\alpha^3}\right)$$

$$\Rightarrow \int_0^\infty x^3 e^{-\alpha x}\,dx = \frac{1.2.3}{\alpha^4} = \frac{3!}{\alpha^4}$$

In general, $\displaystyle\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$

Putting $\alpha = 1$ and $n = z-1$

$$\int_0^\infty x^{z-1} e^{-x}\,dx = (z-1)!$$

We did it !

Problem 2 : Now prove the relation $\Gamma(z+1) = z\Gamma(z)$ using the definite integral definition of gamma function.

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\,dt$$

Replacing z by $z+1$

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^{z}\,dt$$

Integrating by parts -

$$\Gamma(z+1) = t^z \int e^{-t}\,dt\,\Big|_0^\infty - \int_0^\infty \left\{\left(-e^{-t}\right)\frac{d}{dt}\left(t^z\right)\right\} dt$$

$$= -t^z e^{-t}\Big|_0^\infty + \int_0^\infty z t^{z-1} e^{-t}\, dt$$

$$= (0-0) + z\int_0^\infty e^{-t} t^{z-1}\, dt$$

$$= z\,\Gamma(z)$$

We did it again!

Problem 3 : Express the following integrals as Γ functions and evaluate them whenever possible.

(i) $$\int_0^\infty x^{2/3} e^{-x}\, dx = \int_0^\infty x^{5/3-1} e^{-x}\, dx$$

$$= \Gamma(5/3)$$

(ii) $$\int_0^\infty x^{-1/2} e^{-x}\, dx = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, dx$$

$$= \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

(iii) $$\int_0^\infty \sqrt{x}\, e^{-x}\, dx = \int_0^\infty x^{3/2-1} e^{-x}\, dx$$

$$= \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

(iv) $\int_0^\infty x^2 e^{-x^2}\,dx$

Let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$\int_0^\infty t\, e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty t^{\frac{1}{2}} e^{-t}\,dt$$

$$= \frac{1}{2}\int_0^\infty t^{3/2-1} e^{-t}\,dt$$

$$= \frac{1}{2}\,\Gamma(3/2)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}$$

(v) $\int_0^\infty x e^{-x^3}\,dx$

Let, $x^3 = t$

$\Rightarrow 3x^2\,dx = dt$

$\Rightarrow dx = \frac{dt}{3x^2} = \frac{dt}{3t^{2/3}}$

$$\therefore \int_0^\infty x e^{-x^3} dx = \int_0^\infty t^{1/3} e^{-t} \frac{dt}{3t^{2/3}}$$

$$= \frac{1}{3} \int_0^\infty t^{1/3 - 2/3} e^{-t} dt$$

$$= \frac{1}{3} \int_0^\infty t^{-1/3} e^{-t} dt$$

$$= \frac{1}{3} \int_0^\infty t^{2/3 - 1} e^{-t} dt$$

$$= \frac{1}{3} \Gamma(2/3)$$

(vi) $\displaystyle\int_0^1 x^2 \left(\ln \frac{1}{x}\right)^3 dx$

Let $x = e^{-m}$

$\Rightarrow \ln x = -m$

$\Rightarrow \ln\left(\frac{1}{x}\right) = m$

$\Rightarrow \frac{1}{1/x}\left(-\frac{1}{x^2}\right) dx = dm$

$\Rightarrow -\frac{1}{x} dx = dm$

$\Rightarrow dx = -x\, dm$

$= -e^{-m}\, dm$

$$\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx = \int_{\infty}^{0} e^{-2m} m^3 \left(-e^{-m}\,dm\right)$$

$$= \int_0^{\infty} e^{-3m} m^3\,dm$$

$$= \frac{1}{3^4}\int_0^{\infty} e^{-3m}(3m)^3\,d(3m)$$

$$= \frac{1}{81}\Gamma(4)$$

$$= \frac{6}{81} = \frac{2}{27}$$