Learning Objectives

- Periodic Functions
- Orthogonality of Sine and Cosine Function
- Dirichlet Conditions
- Expansion of periodic functions in series of sine & cosine functions and determination of Fourier Coefficients.
- Complex Representation of Fourier Series
- Expansion of Functions with arbitrary period.
- Application to square and triangular waves

Periodic Functions

A periodic function is a function that repeats itself at regular intervals.

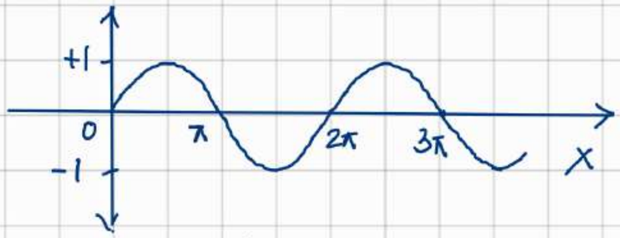
It is expressed as

$$f(x+p) = f(x)$$

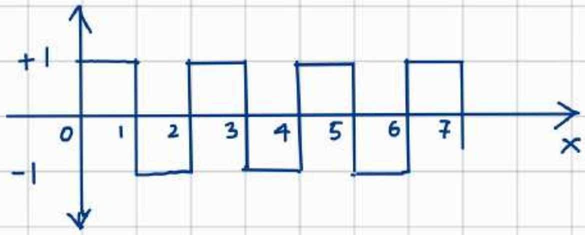
where p is a positive real number and is called the period of the function.

The fundamental period of the periodic function f is the least value of p for which the above expression is true.

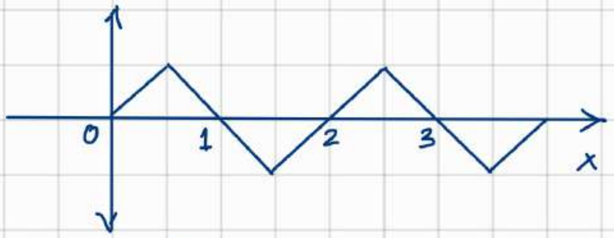
Examples of Periodic Functions.



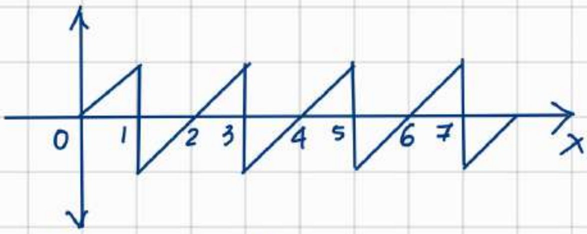
sine function
period $= 2\pi$



square wave
period $= 2$



triangular wave
period $= 2$



sawtooth wave
period $= 2$

Notice that a periodic function's domain is the entire real number line and its range is a specific interval.

$$f : \mathbb{R} \to [a, b]$$

For eg: for sine function, domain $\to \mathbb{R}$
range $\to [-1, +1]$

Points to remember while determining periods

1. If p is the period of the periodic function $f(x)$, then $1/f(x)$ is also a periodic function and will have the same fundamental period p as $f(x)$.

i.e. if $f(x+p) = f(x)$.

then $F(x+p) = F(x)$ where $F(x) = \dfrac{1}{f(x)}$

For eg: just like the $\sin(x)$ function, the $\operatorname{cosec}(x)$ $\left[\text{or } \dfrac{1}{\sin(x)}\right]$ function also has period 2π.

2. If p is the period of the periodic function $f(x)$, then $f(ax+b)$, $a>0$ is also a periodic function with a period of $\dfrac{p}{|a|}$.

For eg: period of $\sin(x) \rightarrow 2\pi$

period of $\sin(2x+1) \rightarrow \frac{2\pi}{2} = \pi$

3. If p is the period of the periodic function $f(x)$, then $af(x)+b$, $a>0$ is also a periodic function with a period p.

For eg: period of $\sin(x) \rightarrow 2\pi$

period of $2\sin(x)+3 \rightarrow 2\pi$.

Example 1: Find the period of $\cos(5x+4)$

period of $\cos(x) \rightarrow 2\pi$

period of $\cos(5x+4) = \frac{2\pi}{|5|} = \frac{2}{5}\pi$

Note: the sum of two periodic function may not always be periodic.

Orthogonality of sine and cosine function

Let's perform some integrals within the limits 0 and 2π (since 2π is the period of sine and cosine function)

1. $$\int_0^{2\pi} \sin(nx)\,dx = \left.\frac{-\cos(nx)}{n}\right|_0^{2\pi} = \frac{1}{n}\left\{-\cos(2\pi n) + \cos 0\right\}$$
$$= 0$$

2. $$\int_0^{2\pi} \cos(mx)\,dx = \left.\frac{\sin(mx)}{m}\right|_0^{2\pi} = \frac{1}{m}\left\{\sin(2m\pi) - \sin(0)\right\}$$
$$= 0$$

3. $$\int_0^{2\pi} \sin^2(nx)\,dx = \frac{1}{2}\int_0^{2\pi} 2\sin^2(nx)\,dx$$
$$= \frac{1}{2}\int_0^{2\pi}\left(1-\cos(2nx)\right)dx$$
$$= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_0^{2\pi}$$
$$= \frac{1}{2}\left[2\pi - 0\right]$$
$$= \pi$$

4. $$\int_0^{2\pi} \cos^2(mx)\,dx = \frac{1}{2}\int_0^{2\pi} 2\cos^2(mx)\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1+\cos(2mx)\right)dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2mx}{2n}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi - 0\right]$$

$$= \pi$$

5. $$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi} 2\sin(mx)\cos(nx)\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}\left\{\sin((m+n)x) + \sin((m-n)x)\right\}dx$$

$$= \frac{1}{2}\left\{\left[-\frac{\cos(m+n)x}{m+n}\right]_0^{2\pi} + \left[-\frac{\cos(m-n)x}{m-n}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{m+n}(1-1) - \frac{1}{m-n}(1-1)\right\}$$

$$= 0$$

6. $\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = ?$ 7 $\int_0^{2\pi} \cos(mx)\cos(nx) = ?$

Hmmm .... $\frac{1}{\pi} \int_0^{2\pi} \sin^2(nx) = 1$

$$\frac{1}{\pi} \int_0^{2\pi} \cos^2(mx) = 1$$

and $\frac{1}{\pi} \int_0^{2\pi} \sin(nx)\cos(mx) = 0$

Can you find any resemblence to something you are familaiar with ?

$$\hat{i} \cdot \hat{i} = 1 \qquad \hat{j} \cdot \hat{j} = 1$$

$$\text{and } \hat{i} \cdot \hat{j} = 0$$

In a way, the definite integral of the sine and cosine functions look like the dot product of basis vectors !

Well, the above relations are called the orthogonality relations of the sine and the cosine functions.

Fourier Series

A periodic function $f(x)$ can be expressed in a Fourier expansion series as :

$$f(x) = \frac{a_0}{2} + \left\{ a_1 \cos(x) + a_2 \cos(2x) + \dots \right\}$$
$$+ \left\{ b_1 \sin(x) + b_2 \sin(2x) + \dots \right\}$$

$$= \frac{a_0}{2} + \sum a_n \cos(nx) + \sum b_n \sin(nx)$$

where $n = 1, 2, 3, \dots$

subject to the Dirichlet Conditions :

1. $f(x)$ should have a finite number of maxima and finite number of minima over the period.
2. $f(x)$ should have a finite number of discontinuities over the period.
3. $f(x)$ should be absolutely integrable over the range of time period.

How to find the Fourier coefficients a_0, a_n and b_n?

$$f(x) = \frac{a_0}{2} + \sum a_n \cos(nx) + \sum b_n \sin(nx)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum a_n \cancelto{0}{\int_{-\pi}^{\pi} \cos(nx)\,dx} + \sum b_n \cancelto{0}{\int_{-\pi}^{\pi} \sin(nx)\,dx}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\left[x\right]_{-\pi}^{\pi}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\,2\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Again,

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{a_0}{2}\cancelto{0}{\int_{-\pi}^{\pi} \cos(mx)\,dx}$$

$$+ \sum_n a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$$

$$+ \sum_n b_n \cancelto{0}{\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx}$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \sum_n a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$$

$$= a_1 \int_{-\pi}^{\pi} \cancelto{0}{\cos(mx)\cos(x)}\,dx \; +$$

$$a_2 \int_{-\pi}^{\pi} \cancelto{0}{\cos(mx)\cos(2x)}\,dx + \ldots$$

$$\ldots + a_m \int_{-\pi}^{\pi} \cancelto{\pi}{\cos(mx)\cos(mx)}\,dx + \ldots$$

$$= a_m \pi$$

hence $$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}$$

Similarly,

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cancelto{0}{\sin(mx)}\,dx$$

$$+ \sum_n a_n \int_{-\pi}^{\pi} \cancelto{0}{\cos(nx)\sin(mx)}\,dx$$

$$+ \sum_n b_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \sum_n b_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$$

$$= b_1 \int_{-\pi}^{\pi} \sin(x)\sin(mx)\,dx \xrightarrow{} 0 \quad +$$

$$b_2 \int_{-\pi}^{\pi} \sin(2x)\sin(mx)\,dx \xrightarrow{} 0 \quad + \ldots$$

$$\ldots + b_m \int_{-\pi}^{\pi} \sin(mx)\sin(mx)\,dx \xrightarrow{} \pi \quad + \ldots$$

$$= b_m \pi$$

hence, $$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Tip : 1. if $f(x)$ is an odd function,

$a_0 = a_n = 0$; only b_n survives

2. if $f(x)$ is an even function,

$b_n = 0$; a_0 & a_n survives.

Example 1 : $f(x) = x \quad ; \quad -\pi < x < \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx$$

$$= \frac{1}{\pi} \left[x \frac{1}{n} \sin(nx) - \int \frac{1}{n} \sin(nx)\, dx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} \left[\frac{\cos(nx)}{n} \right]_{-\pi}^{\pi} \right\}$$

$$= \frac{1}{\pi n^2} \left\{ -1 - (-1) \right\}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx$$

$$= \frac{1}{\pi}\left[x \int \sin(nx)\,dx - \int\left(\int \sin(nx)\,dx\right)dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(nx)}{n}\right) - \int\left(-\frac{\cos(nx)}{n}\right)dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-\pi)(-1)}{n} - \frac{(-\pi)(-1)}{n}\right]$$

$$= \frac{2\pi}{\pi n}$$

$$= \frac{2}{n}$$

Thus, $a_0 = a_n = 0$

$$b_n = \frac{2}{n}$$

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$$

$$= 2\sin(x) + \sin(2x) + \frac{2}{3}\sin(3x) + \dots$$

Example 2 : $f(x) = x^2 \quad ; \quad -\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\left[\pi^3 - (-\pi^3)\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx$$

$$= \frac{1}{\pi}\left[x^2\int \cos(nx)\, dx - \int\left(2x\int \cos(nx)\, dx\right) dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\underbrace{x^2\,\frac{\sin(nx)}{n}}_{\to\, 0 \text{ at } \pi \,\&\, -\pi} - 2\int x\,\frac{\sin(nx)}{n}\, dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2}{n^2}x\cos(nx) + \frac{2}{n}\int \frac{-\cos(nx)}{n}\, dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2}{n^2}\pi - \frac{2}{n^2}\pi - \frac{2}{n^2}\,\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi}$$

$$= -\frac{4}{n^2}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos(nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2}\right) \cos(nx)$$

Change of interval : Expansion of functions with arbitrary period $2l$.

For a function with periodicity 2π it can be expressed as

$$f(x) = \frac{a_0}{2} + \sum a_n \cos(nx) + \sum b_n \sin(nx)$$

For a function with periodicity $2l$, it can be expressed in a Fourier Series as -

$$\boxed{f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\left(\frac{n\pi x}{l}\right)}$$

The fourier coefficients are :-

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Complex Notation of Fourier Series

$$f(x) = \frac{a_0}{2} + \sum a_n \cos(nx) + \sum b_n \sin(nx)$$

$$= \frac{a_0}{2} + \sum a_n \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$$

$$+ \sum b_n \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$= \frac{a_0}{2} + \sum \frac{1}{2}\left(a_n + \frac{b_n}{i}\right) e^{inx} +$$

$$\sum \frac{1}{2}\left(a_n - \frac{b_n}{i}\right) e^{-inx}$$

$$= \frac{a_0}{2} + \sum \frac{1}{2}\left(a_n - ib_n\right) e^{inx} +$$

$$\sum \frac{1}{2}\left(a_n + ib_n\right) e^{-inx}$$

$$= \frac{a_0}{2} + \sum A_n e^{inx} + \sum B_n e^{-inx}$$

$n = 1, 2, 3, \cdots$

Combining we may write

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}}$$

How to find the Fourier coefficient C_n ?

For positive n,

$$C_n = \frac{1}{2}\left(a_n - ib_n\right)$$

$$= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(x)\left(\cos(nx) - i\sin(nx)\right)dx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

Thus, a periodic function $f(x)$ with periodicity 2π may be expressed in a Fourier series as.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{inx}$$

Where. $$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

For a function with arbitrary period $2l$,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\frac{\pi}{l}x}$$

and $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\frac{\pi}{l}x}\, dx$$