A second order linear differential equation has the general form :

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x).$$

For homogeneous equation with constant coefficients,

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0$$

The simplest way to solve such an equation is to assume a solution of the form $y = e^{\lambda x}$.

Example 1 :- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Let's assume a solution $y = e^{\lambda x}$

$$\frac{d^2}{dx^2}\left(e^{\lambda x}\right) - 3\frac{d}{dx}\left(e^{\lambda x}\right) + 2e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-1) = 0$$

$$\lambda = 2 \ ; \ \lambda = 1.$$

Thus, $y_1 = e^x$ and $y_2 = e^{2x}$

We shall always check whether the solutions satisfy the equation.

for $y_1 = e^x$

$$LHS = \frac{d^2}{dx^2}(e^x) - 3\frac{d}{dx}(e^x) + 2e^x$$

$$= e^x - 3e^x + 2e^x$$

$$= 0$$

$$= RHS$$

for $y_2 = e^{2x}$

$$LHS = \frac{d^2}{dx^2}(e^{2x}) - 3\frac{d}{dx}(e^{2x}) + 2e^{2x}$$

$$= 4e^{2x} - 6e^{2x} + 2e^{2x}$$

$$= 0$$

$$= RHS$$

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

Thus, $y_1 = e^x$ and $y_2 = e^{2x}$ are the solutions of the given equation.

Example 2 : Let's do the simple harmonic oscillator equation :-

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

Assuming the solution $y = e^{\lambda x}$,

$$\frac{d^2}{dx^2}\left(e^{\lambda x}\right) + \omega^2 e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 e^{\lambda x} + \omega^2 e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 + \omega^2 = 0$$

$$\Rightarrow \lambda = \pm i\omega$$

Thus, $y_1 = e^{i\omega x}$ and $y_2 = e^{-i\omega x}$

* Check for yourself whether these are independent solutions and also that they indeed satisfy the equation.

Let's modify the RHS of Example 1 to make it an inhomogenous equation :

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 1$$

Well, the solution to an inhomogenous 2^{nd} order differential equation may be assumed to be composed of two parts :

$$y = y_c + y_p$$

where y_c is called the complementary function

and y_p is called the particular integral

How to find the complementary function y_c ?

Step I : make the equation homogenous by making $f(x) = 0$

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = f(x) = 0$$

Step II : assume a solution of the form $y = e^{\lambda x}$ for the equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0$$

Step III : Find λ and check the nature of λ.

(i) if all λ are real and distinct, then assume y_c to be

$$y_c = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

(ii) if all λ are complex and distinct, then assume y_c to be :

$$y_c = C_1 e^{\overbrace{(\alpha + i\beta)}^{\lambda_1} x} + C_2 e^{\overbrace{(\alpha - i\beta)}^{\lambda_2} x}$$

(iii) if both λ are equal, then we assume y_c to be :

$$y_c = (C_1 + C_1 x) e^{\lambda x}$$

How to find the particular integral y_p ?

Depending on the form of $f(x)$ we assume the form of y_p as follows :

(i) if $f(x) = Ae^{mx}$, then y_p is assumed to be

$$y_p = Be^{mx}$$

(ii) if $f(x) = A_1 \sin(mx) + A_2 \cos(mx)$, then y_p is assumed to be

$$y_p = B_1 \sin(mx) + B_2 \cos(mx)$$

(iii) if $f(x) = A_0 + A_1 x + A_2 x^2 + \ldots$ then y_p is assumed to be

$$y_p = B_0 + B_1 x + B_2 x^2 + \ldots$$

(iv) if $f(x)$ is the sum or product of any of the above function, then the assumed solution will be the product and sum of the corresponding assumed solutions.

Example 3 : $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 1$$

From Example 1 we already found that $\lambda = 2$ and 1 by assuming a solution $y = e^{\lambda x}$ for $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

Hence, $y_c = Ae^{x} + Be^{2x}$

Since $f(x) = 1$; $\therefore y_p = C$

So, we assume the solution $y = Ae^{x} + Be^{2x} + C$

Now, $$\frac{dy}{dx} = Ae^{x} + 2Be^{2x}$$

$$\frac{d^2y}{dx^2} = Ae^{x} + 4Be^{2x}$$

The original equation now becomes

$$\left(Ae^{x}+4Be^{2x}\right)-3\left(Ae^{x}+2Be^{2x}\right)$$
$$+2\left(Ae^{x}+Be^{2x}+C\right)=1$$

$\Rightarrow 2C=1$

$\Rightarrow C=\frac{1}{2}$

Thus, the solution is

$$y=Ae^{x}+Be^{2x}+\frac{1}{2}$$

* Check whether any arbitrary value of A and B satisfies the equation or we need to find specific values for the coefficients A and B.

Example 4 : $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{3x}$

$$y_c=Ae^{x}+Be^{2x}$$

$$y_p=Ce^{3x}$$

$$y=Ae^{x}+Be^{2x}+Ce^{3x}$$

$$\frac{dy}{dx}=Ae^{x}+2Be^{2x}+3Ce^{3x}$$

$$\frac{d^2y}{dx^2}=Ae^{x}+4Be^{2x}+9Ce^{3x}$$

putting the above in the original equation,

$$\left(Ae^{x} + 4Be^{2x} + 9Ce^{3x}\right) - 3\left(Ae^{x} + 2Be^{2x} + 3Ce^{3x}\right)$$
$$+ 2\left(Ae^{x} + Be^{2x} + Ce^{3x}\right) = e^{3x}$$

$$\Rightarrow \quad 2Ce^{3x} = e^{3x}$$

$$\Rightarrow \quad C = \frac{1}{2}$$

Thus, the solution is $y = Ae^{x} + Be^{2x} + \frac{1}{2}e^{3x}$

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Example 5 : $\quad \dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \sin(x)$

$$y_c = Ae^{x} + Be^{2x}$$
$$y_p = C\sin(x) + D\cos(x)$$
$$y = Ae^{x} + Be^{2x} + C\sin(x) + D\cos(x)$$
$$\frac{dy}{dx} = Ae^{x} + 2Be^{2x} + C\cos(x) - D\sin(x)$$
$$\frac{d^2y}{dx^2} = Ae^{x} + 4Be^{2x} - C\sin(x) - D\cos(x)$$

putting the above in the original equation

$$\left(Ae^{x} + 4Be^{2x} - C\sin(x) - D\cos(x)\right) -$$
$$3\left(Ae^{x} + 2Be^{2x} + C\cos(x) - D\sin(x)\right) +$$

$$2\left(Ae^{x} + Be^{2x} + C\sin(x) + D\cos(x)\right) = \sin(x)$$

$$\Rightarrow \left(-C + 3D + 2C\right)\sin(x) + \left(-D - 3C + 2D\right)\cos(x) = \sin(x)$$

$$\Rightarrow (C + 3D)\sin(x) + (D - 3C)\cos(x) = \sin(x)$$

Equating the coefficients of $\sin(x)$ and $\cos(x)$

$D - 3C = 0$ and $C + 3D = 1$

$\Rightarrow D = 3C$

$\Rightarrow C + 3(3C) = 1$

$\Rightarrow 10C = 1$

$\Rightarrow C = \frac{1}{10}$

Thus, $C = \frac{1}{10}$ and $D = \frac{3}{10}$

Thus, the solution is

$$y = Ae^{x} + Be^{2x} + \frac{1}{10}\sin(x) + \frac{3}{10}\cos(x)$$

* Check if the solution indeed satisfies the given equation.

Example 6 : $y'' - y = e^{-x}$

for $y'' - y = 0$ let us assume a solution

$$y = e^{\lambda x}.$$

$$y'' - y = 0$$

$$\Rightarrow \lambda^2 e^{\lambda x} - e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

$$y_1 = e^x \quad ; \quad y_2 = e^{-x}$$

Thus, $y_c = Ae^x + Be^{-x}$

since we already have e^{-x} in y_c
so we assume the form of y_p as

$$y_p = Cxe^{-x}$$

Hence, assumed solution is :

$$y = y_c + y_p$$
$$= Ae^x + Be^{-x} + Cxe^{-x}$$

$$y' = Ae^{x} - Be^{-x} + Ce^{-x} - Cxe^{-x}$$

$$y'' = Ae^{x} + Be^{-x} - Ce^{-x} - Ce^{-x} + Cxe^{-x}$$

substituting in the original equation,

$$Ae^{x} + Be^{-x} - 2Ce^{-x} + Cxe^{-x} - Ae^{x} - Be^{-x} - Cxe^{-x} = e^{-x}$$

$$\Rightarrow \quad -2Ce^{-x} = e^{-x}$$

$$\Rightarrow \quad C = -\frac{1}{2}$$

$$y = Ae^{x} + Be^{-x} - \frac{1}{2}e^{-x}$$

* Check if the solution indeed satisfies the given equation.

Example 7 :- $y'' + 4y = x^{2}\sin(2x)$

for $y'' + 4y = 0$ let us assume a solution

$$y = e^{\lambda x}$$

$$y'' + 4y = 0$$

$$\Rightarrow \quad \lambda^{2}e^{\lambda x} + 4e^{\lambda x} = 0$$

$$\Rightarrow \quad \lambda^{2} + 4 = 0$$

$$\Rightarrow \quad \lambda = \pm 2i$$

$$\begin{aligned} y_c &= ae^{2ix} + be^{-2ix} \\ &= a\cos(2x) + ia\sin(2x) + b\cos(2x) \\ &\qquad - ib\sin(2x) \\ &= (a+b)\cos(2x) + (ia - ib)\sin(2x) \\ &= A\cos(2x) + B\sin(2x) \end{aligned}$$

$$y_p = (a + bx + cx^2)\{Cx\sin(2x) + Dx\sin(2x)\}$$

$$\begin{aligned} y &= y_c + y_p \\ &= A\cos(2x) + B\sin(2x) + \\ &\qquad (a+bx+cx^2)\{Cx\sin(2x) + Dx\sin(2x)\} \end{aligned}$$

* Complete the steps and check if the solution indeed satisfies the given equation.

We see that this method works pretty well for 2nd order inhomogenous ODE with constant coefficients :

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

What if we consider that the coefficients are not constant ?

i.e. $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = f(x)$$

—

Let's try the simple case of

$$y'' - 4xy = 0$$

Let's assume the solution $y = e^{\lambda x}$

$$y'' - 4xy = 0$$

$$\Rightarrow \lambda^2 e^{\lambda x} - 4x e^{\lambda x} = 0$$

$$\Rightarrow \lambda = \pm 2\sqrt{x}$$

$$y_1 = e^{2x^{3/2}}$$

$$y_1' = 2e^{2x^{3/2}} \cdot x^{1/2}\, 3/2$$

$$= 3e^{2x^{3/2}} x^{1/2}$$

$$y_2 = e^{-2x^{3/2}}$$

$$y'' = 3\left[e^{2x^{3/2}} \cdot \frac{1}{2\sqrt{x}} + x^{1/2}\, 3e^{2x^{3/2}} \cdot x^{1/2}\right]$$

$$= 3\left(e^{2x^{3/2}} \frac{1}{2\sqrt{x}} + 3x e^{2x^{3/2}}\right)$$

Here, LHS $\neq$ RHS

We see that this method is not suitable for a general 2^{nd} order ODE.

This leads us to the infinite series solution method for second order ODE.

Series solution / Forbenius Method

Any function may be expressed in terms of the Taylor series expansion –

$$y(x) = y(x)\Big|_{x=x_0} + (x-x_0)\, y^{(1)}(x)\Big|_{x=x_0} +$$

$$\frac{1}{2!}(x-x_0)^2 y^{(2)}(x)\Big|_{x=x_0} + \frac{1}{3!}(x-x_0)^3 y^{(3)}(x)\Big|_{x=x_0}$$

$$+ \ldots\ldots$$

$$= \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Let's try to solve : $y'' - 4xy = 0$

We assume the solution

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$$

$$y' = \sum_{\lambda=0}^{\infty} a_\lambda \lambda x^{\lambda-1}$$

$$y'' = \sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2}$$

Now, $y'' - 4xy = 0$

$$\Rightarrow \sum_{\lambda=0}^{\infty} \lambda(\lambda-1) a_\lambda x^{\lambda-2} - 4x \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} \lambda(\lambda-1) a_\lambda x^{\lambda-2} - 4 \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+1} = 0$$

$$\Rightarrow \Big(0+0+2(2-1)a_2 x^{2-2} + 3(3-1)a_3 x^{3-2} + 4(4-1)a_4 x^{4-2} + \dots\Big) - 4\left(a_0 x + a_1 x^2 + a_2 x^3 + \dots\right) = 0$$

$$\Rightarrow 2a_2 + (6a_3 - 4a_0)x + (12a_4 - 4a_1)x^2 + \dots = 0$$

Comparing the coefficients of x^n,

$2a_2 = 0$	$6a_3 - 4a_0 = 0$	$12a_4 - 4a_1 = 0$
$\Rightarrow a_2 = 0$	$\Rightarrow 2a_0 = 3a_3$	$\Rightarrow a_1 = 3a_1$
	$\Rightarrow a_0 = \frac{3}{2} a_3$	

We need to find the recurrence relation -

$$\sum a_\lambda \lambda(\lambda-1) x^{\lambda-2} - 4 \sum a_\lambda x^{\lambda+1} = 0$$

Comparing the coefficients of the lowest power of x,

$$a_2 = 0 \quad ;$$

Comparing the coefficients of the general power of x, we get the recurrence relation,

$$a_{j+3}(j+3)(j+2)x^{j+1} - 4a_j x^{j+1} = 0$$

$$\Rightarrow a_{j+3} = \frac{4}{(j+2)(j+3)} a_j$$

$$a_2 = 0$$

$j = 0$, $\quad a_3 = \frac{4}{2 \cdot 6} a_0 = \frac{1}{3} a_0$

$j = 1$, $\quad a_4 = \frac{4}{3 \cdot 4} a_1 = \frac{1}{3} a_1$

$$j=2\ , \quad a_5 = \frac{4}{4.5} a_2 = 0$$

$$j=3\ , \quad a_6 = \frac{4}{5.6} a_3 = \frac{2}{15} \cdot \frac{1}{3} a_0 = \frac{2}{45} a_0$$

$$j=4\ , \quad a_7 = \frac{4}{6.7} a_4 = \frac{2}{21} \cdot \frac{1}{3} a_1 = \frac{2}{63} a_1$$

$$\cdots$$

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + 0 + \frac{1}{3} a_0 x^3 + \frac{1}{3} a_1 x^4 +$$

$$+ 0 + \frac{2}{45} a_0 x^6 + \frac{2}{63} a_1 x^7 + \cdots$$

$$= a_0 \left(1 + \frac{1}{3} x^3 + \frac{2}{45} x^6 + \cdots \right) +$$

$$a_1 \left(x + \frac{1}{3} x^4 + \frac{2}{63} x^7 + \cdots \right)$$

Example 8 : $y'' + x^2 y = 0$

Let us assume the solution

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$$

$$y' = \sum_{\lambda=0}^{\infty} a_\lambda \lambda x^{\lambda-1}$$

$$y'' = \sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2}$$

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2} + x^2 \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+2} = 0$$

Equating the coefficients of x^0;

$$a_2 2(2-1) x^{2-2} = 0$$

$$\Rightarrow 2a_2 = 0 \Rightarrow a_2 = 0$$

Equating the coefficients of x^1;

$$a_3 3(3-1) x^{3-2} = 0$$

$$\Rightarrow 6a_3 = 0 \Rightarrow a_3 = 0$$

Equating the coefficients of any general power of x for,

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+2} = 0$$

$$a_{j+4} (j+4)(j+3) x^{j+4-2} + a_j x^{j+2} = 0$$

$$\Rightarrow a_{j+4} = -\frac{1}{(j+3)(j+4)} a_j$$

for $j=0$, $a_4 = -\frac{1}{12} a_0$

$j=1$, $a_5 = -\frac{1}{20} a_1$

$j=2$, $a_6 = -\frac{1}{30} a_2 = 0$

$j=3$, $a_7 = -\frac{1}{42} a_3 = 0$

$j=4$, $a_8 = -\frac{1}{56} a_4 = \left(-\frac{1}{56}\right)\left(-\frac{1}{12}\right) a_0 = \frac{1}{672} a_0$

$j=5$, $a_9 = -\frac{1}{72} a_5 = \left(-\frac{1}{72}\right)\left(-\frac{1}{20}\right) a_1 = \frac{1}{1440} a_1$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$+ a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8$$

$$+ a_9 x^9 + a_{10} x^{10} + \cdots$$

$$= a_0 + a_1 x + 0 + 0 - \frac{1}{12} a_0 x^4$$

$$- \frac{1}{120} a_1 x^5 + 0 + 0 + \frac{1}{672} a_0 x^8$$

$$+ \frac{1}{1440} a_1 x^9 + \cdots$$

$$= \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 - \cdots\right) a_0 +$$

$$\left(x - \frac{1}{120} x^5 + \frac{1}{1440} x^9 - \cdots\right) a_1$$

Example 9: $y'' + \omega^2 y = 0$

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

$$y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$$

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

Equating the coefficients of the lowest power of x,

$$a_0 k(k-1) = 0$$

$$\Rightarrow \boxed{k(k-1) = 0} \qquad \therefore k=0 \text{ and } k=1$$

indicial eq^n

For $k = 0$;

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda} = 0$$

Equating the coefficients of the lowest power of x

for $\lambda = 0$; $a_0 x^0 = 0$ but $a_0 \neq 0$

for $\lambda = 1$; $a_1 x^1 = 0 \Rightarrow a_1 = 0$

This gives $a_{j+2}(j+2)(j+1)\,x^j + \omega^2 a_j x^j = 0$

$$\Rightarrow a_{j+2} = -\frac{\omega^2}{(j+2)(j+1)}\,a_j$$

for $j=0$; $\quad a_2 = -\dfrac{\omega^2}{2.1}\,a_0$

for $j=2$; $\quad a_4 = -\dfrac{\omega^2}{4.3}\,a_2 = +\dfrac{\omega^4}{4.3.2.1}\,a_0$

for $j=4$; $\quad a_6 = -\dfrac{\omega^2}{6.5}\,a_4 = -\dfrac{\omega^6}{6.5.4.3.2.1}\,a_0$

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upon inspection we find that :

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!}\,a_0$$

$\therefore$ the solution is

$$y = a_0 + \cancel{a_1 x}^{\,0} + a_2 x^2 + \cancel{a_3 x^3}^{\,0} + a_4 x^4 + \dots$$

$$= a_0 - \frac{\omega^2}{2!}\,a_0 x^2 + \frac{\omega^4}{4!}\,a_0 \omega^4 - \frac{\omega^6}{6!}\,a_0 \omega^6 + \dots$$

$$= a_0\left(1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \dots\right)$$

$$= a_0 \cos(\omega x)$$

For $k = 1$;

$$\sum_{\lambda=0}^{\infty} a_\lambda (1+\lambda)(1+\lambda-1) x^{1+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{1+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (1+\lambda)\lambda x^{\lambda-1} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{1+\lambda} = 0$$

Equating the coefficients of the lowest power of x

$$a_1 \, 2.1. x^0 = 0$$

$$\Rightarrow a_1 = 0$$

Equating the coefficients of the general power of x

$$a_{j+2} (j+3)(j+2) x^{j+1} + \omega^2 a_j x^{j+1} = 0$$

$$\Rightarrow a_{j+2} = - \frac{\omega^2}{(j+2)(j+3)} a_j$$

For $j=0$, $a_2 = - \dfrac{\omega^2}{2.3} a_0$

For $j=2$, $a_4 = - \dfrac{\omega^2}{4.5} a_2 = + \dfrac{\omega^4}{2.3.4.5} a_0$

For $j=4$, $a_6 = - \dfrac{\omega^2}{6.7} a_4 = - \dfrac{\omega^6}{2.3.4.5.6.7} a_0$

. . . . .

upon inspection we find that

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

$$\begin{aligned}
y &= a_0 x + \cancelto{0}{a_1} x^2 + a_2 x^3 + \cancelto{0}{a_3} x^4 + a_4 x^5 + \dots \\
&= a_0 x - \frac{\omega^2}{3!} a_0 x^3 + \frac{\omega^4}{5!} a_0 x^5 - \dots \\
&= \frac{a_0}{\omega}\left(\omega x - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \dots \right) \\
&= \frac{a_0}{\omega} \sin(\omega x)
\end{aligned}$$

Example 10 : $y'' + xy' + 2y = 0$

We assume the solution

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \; ; \quad a_0 \neq 0$$

$$\therefore \quad y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$$

Substituting the above into our original equation,

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + x \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$
$$+ 2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda \{k+\lambda+2\} x^{k+\lambda} = 0$$

Equating the coefficients of the lowest power of x

for $\lambda = 0$, $a_0 k(k-1) = 0$

Since, $a_0 \neq 0$, therfore $k = 0$ and $k = 1$

For $k = 0$, we get :

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda (\lambda+2) x^{\lambda} = 0$$

Equating the coefficient of the lowest power of x.

$$a_0 (0+2) x^0 = 0$$

$$\Rightarrow a_0 = 0$$

but we considered $a_0 \neq 0$.

Hence, $a_1 (1+2) x^1 = 0$

$$\Rightarrow a_1 = 0$$

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Equating the coefficients of the general power of x

$$a_{j+2} (j+2)(j+1) x^j + a_j (j+2) x^j = 0$$

$$\Rightarrow a_{j+2} = -\frac{a_j}{j+1}$$

For $j = 0$, $a_2 = -a_0$

For $j = 1$, $a_3 = -\frac{a_1}{2} = 0$

For $j = 2$, $a_4 = -\frac{a_2}{3} = \frac{a_0}{3}$

For $j = 3$, $a_5 = 0$

For $j = 4$, $a_6 = -\frac{a_4}{5} = -\frac{a_0}{3.5}$

$$----$$

upon inspection we find that.

$$a_{2n} = (-1)^n \frac{1}{1.3.5..(2n-1)} a_0$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + 0 + (-a_0 x^2) + 0 + \frac{a_0}{3} x^4 + 0 - \frac{a_0}{3.5} x^6 + \cdots$$

$$= a_0 \left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15} + \cdots \right)$$

Again for $k = 1$, we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (1+\lambda)(1+\lambda-1) x^{1+\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda \{1+\lambda+2\} x^{1+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (1+\lambda)\lambda x^{\lambda-1} + \sum_{\lambda=0}^{\infty} a_\lambda (\lambda+3) x^{\lambda+1} = 0$$

Equating the coefficients of the lowest power of x.

$$a_1 \, 2 \cdot 1 \, x^0 = 0$$

$$\Rightarrow a_1 = 0$$

Equating the coefficients of the general power of x,

$$a_{j+2}(j+3)(j+2)x^{j+2-1} + a_j(j+3)x^{j+1} = 0$$

$$\Rightarrow a_{j+2} = -\frac{1}{(j+2)} a_j$$

for $j=0$; $a_2 = -\frac{1}{2} a_0$

for $j=1$; $a_3 = \frac{1}{3} a_1 = 0$

for $j=2$; $a_4 = -\frac{1}{4} a_2 = \frac{1}{8} a_0$

for $j=3$; $a_5 = \frac{1}{5} a_3 = 0$

upon inspection we find that.

$$a_{2n} = (-1)^n \frac{1}{2.4.6\ldots(2n)} a_0$$

Hence, $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{1+\lambda}$

$$= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \ldots$$

$$= a_0 x + 0 + \left(-\frac{1}{2} a_0\right) x^3 + 0 + \left(\frac{1}{8} a_0\right) x^5 + \ldots$$

$$= a_0 \left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5 + \ldots\right)$$

$=$

Thus the complete solution is

$$y = A\left(1 - x^2 + \frac{x^4}{3} - \frac{x6}{15} + \ldots\right)$$

$$+ B\left(x - \frac{1}{2} x^3 + \frac{1}{8} x^5 - \ldots\right)$$

Homework :

Solve the following differential equations using the series solution method :-

1. $x^2y'' - 6y = 0$

2. $x^2y'' + xy' - b^2y = 0$

3. $x^2y'' - xy' + (1-x)y = 0$

4. $4xy'' + 2y' + y = 0$

Ordinary point, Singular point and the Fuchs Theorem

The general equation for a homogenous second order ordinary differential equation is :-

$$y'' + P(x)y' + Q(x)y = 0$$

↳ If P and Q remains finite at $x = x_0$, then x_0 is called an ordinary point.

↳ If P or Q or both diverges at $x = x_0$, then x_0 is called a singular point.

Singular point

→ If P or Q diverges at $x = x_0$ but $(x-x_0)P$ and $(x-x_0)^2 Q$ remains finite then x_0 will be called a regular singular point

→ If P or Q diverges at $x = x_0$ and $(x-x_0)P$ and $(x-x_0)^2 Q$ also diverges at $x = x_0$, then x_0 is called irregular singular point.

While solving the above differential equations, you might have wondered when the series solution method is bound to work and when it is not.

Let me introduce you to the Fuch's theorem :

Fuchs' Theorem :

We can always obtain atleast one power series solution when applying the Forbenius method provided we are expanding about an ordinary point or a regular singular point.

Whether we get one or two distinct solution depends on the root of the indicial equation (i.e. the value of k)

1. If two roots are equal, we obtain only one solution.

2. If two roots differ by a non-integer, no two distinct solutions may be obtained.

3. If the roots differ by an integer, one of the two will yield a solution.