Let me introduce you to a few differential equations and their solutions which are of prime importance to a Physicist.

1. Bessel's Equation

$$x^2 y'' + xy' + (x^2 - n^2) y = 0$$

2. Legendre's Equation

$$(1-x^2) y'' - 2xy' + n(n+1) y = 0$$

3. Hermite's Equation

$$y'' - 2xy' + 2\alpha y = 0$$

4. Laguerre's Equation

$$xy'' + (1-x) y' + \alpha y = 0$$

5. Chebyshev Equation

$$(1-x^2) y'' - xy' + n^2 y = 0$$

In this chapter we shall look at the Legendre, Hermite and the Laguerre Polynomials.

Let us also look at the singular points for these equations -

Equation	Regular singularity	Irregular Singularity
Bessel	0	∞
Legendre	$-1, 1, \infty$	-
Laguerre	0	∞
Hermite	-	∞
Chebyshev	$-1, 1, \infty$	-

We shall take a look at only the Hermite, Laguerre and the Legendre's differential equations in this course.

We are interested not only in solving the equations but also in understanding what the solutions entail.

1. Legendre's Differential Equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

We shall assume a solution of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \quad ; \quad a_0 \neq 0$$

This gives, $y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$

and $y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$

Thus the equation becomes,

$$(1-x^2)\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2}$$

$$-2x\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{k+\lambda-1}$$

$$+ n(n+1)\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda}$$

$$-\sum_{\lambda=0}^{\infty} a_\lambda\, 2(k+\lambda)x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda\, n(n+1)x^{k+\lambda} = 0$$

The indicial equation in this case reads.

$$k(k-1) = 0$$

Thus, $k = 0$ and $k = 1$.

For $k=0$,

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1)x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda \{n(n+1) - \lambda(2+\lambda-1)\} x^\lambda = 0$$

Equating the coefficients of the general power of x, we obtain the following recurrence relation:

$$a_{j+2}(j+2)(j+1)x^{j+2-2} + a_j\{n(n+1) - j(j+1)\}x^j = 0$$

$$\Rightarrow a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} a_j$$

For $j=0$; $\quad a_2 = -\frac{n(n+1)}{1.2} a_0 = -\frac{n(n+1)}{2!} a_0$

$j=2$; $\quad a_4 = \frac{2.3 - n(n+1)}{3.4} a_2$

$$= -\frac{[6 - n(n+1)][n(n+1)]}{1.2.3.4} a_0$$

$$= -\frac{[6 - n^2 - n][n(n+1)]}{1.2.3.4} a_0$$

$$= \frac{[(n-2)(n+3)][n(n+1)]}{1.2.3.4} a_0$$

$$= \frac{n(n+1)(n-2)(n+3)}{4!} a_0$$

Similarly.

$$j = 4 \;;\; a_6 = -\frac{n(n+1)(n-2)(n+3)(n-4)(n+5)}{6!} a_0$$

and so on ..

Thus, the solution is.

$$y_1 = \sum a_\lambda x^{k+\lambda} \qquad (k=0)$$

$$= a_0 + \cancelto{0}{a_1 x} + a_2 x^2 + \cancelto{0}{a_3 x^3} + \dots$$

$$= a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \dots$$

$$= a_0 \Big(1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4$$

$$- \frac{n(n+1)(n-2)(n+3)(n-4)(n+5)}{6!} x^6 + \dots$$

Similarly for $k=1$,

$$\sum_{\lambda=0}^{\infty} a_\lambda (1+\lambda)\lambda\, x^{\lambda-1}$$
$$- \sum_{\lambda=0}^{\infty} a_\lambda \left[2(1+\lambda) + (1+\lambda)\lambda - n(n+1)\right] x^{\lambda+1} = 0$$

Equating the coefficients of the general power of x, we obtain the following recurrence relation.

$$a_{j+2}(1+j+2)(j+2) - a_j\left[2(1+j) + (1+j)j - n(n+1)\right]$$

$$\Rightarrow a_{j+2} = \frac{(j+2)(j+1) - n(n+1)}{(j+2)(j+3)}\, a_j$$

for $j=0$; $a_2 = \dfrac{2 - n(n+1)}{2\cdot 3}\, a_0$

$$= -\frac{n^2+n-2}{3!}\, a_0$$

$$= -\frac{n(n+2)-1(n+2)}{3!}\, a_0$$

$$= -\frac{n(n-1)(n+2)}{3!}\, a_0$$

$$j=2, \quad a_4 = \frac{4.3 - n(n+1)}{4.5} a_2$$

$$= \left(-\frac{n^2+n-4.3}{4.5}\right)\left(-\frac{n(n+1)(n-2)}{3!} a_0\right)$$

$$= \frac{n(n+4)-3(n+4)}{4.5} \cdot \frac{n(n-1)(n+2)}{3!} a_0$$

$$= \frac{n(n-1)(n+2)(n-3)(n+4)}{5!} a_0$$

and so on ...

Thus, the solution is.

$$y_2 = \sum_\lambda a_\lambda x^{k+\lambda} \qquad (k=1)$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \cdots$$

$$= a_0 x - \frac{n(n-1)(n+2)}{3!} a_0 + \frac{n(n-1)(n+2)(n-3)(n+4)}{5!} a_0 + \cdots\cdots$$

the most general solution is $y = y_1 + y_2$.

Both y_1 and y_2 are solutions but if n is even we shall consider y_1 as solution and when n is odd we shall consider y_2 as solution.

Let us try to examine what different values of n yield us :

for $n=0$, $y_0 = \text{const.}$

for $n=1$, $y_1 = a_0 x$

for $n=2$, $y_2 = a_0 \left[1 - \frac{2.3}{2!} x^2\right] = a_0\left(1-3x^2\right)$

for $n=3$, $y_3 = a_0 \left[x - \frac{2.5}{3!} x^3\right] = a_0\left(x - \frac{5}{3} x^3\right)$

for $n=4$, $y_4 = a_0 \left[1 - \frac{4.5}{2!} x^2 + \frac{4.5.2.7}{4!} x^4\right]$

$$= a_0\left(1 - 10x^2 + \frac{35}{3} x^4\right)$$

—

Physicists often like to write these solutions in their normalised form by adjusting the constant a_0.

These solutions are called Legendre polynomials. $P_l(x)$

$P_0(x) = 1$ $\qquad$ $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$ $\qquad$ $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

Generating function

A generating function $G(x,t)$ is a function such that, when expanded as a power series of t, the co-efficients $C_n(x)$ of each term can generate polynomials.

Question : Obtain the Legendre Polynomials from its generating function.

The generating function for Legendre Polynomials is

$$G(x,t) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}}$$

If we expand it as a power series in t the co-efficients will give us the Legendre polynomials $P_l(x)$.

$$\text{i.e.} \quad G(x,t) = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

Now, let us expand the generating function $G(x,t)$ using binomial expansion.

$$G(x,t) = \left(1 - 2xt + t^2\right)^{-1/2}$$

$$= \left(1 - \left(2xt - t^2\right)\right)^{-1/2}$$

$$= 1 - \left(-\tfrac{1}{2}\right)\left(2xt - t^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(2xt - t^2\right)^2$$

$$-\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(2xt-t^2)^3+\cdots$$

$$= 1 + \frac{1}{2}(2xt-t^2) + \frac{1.3}{2^2 2!}(2xt-t^2)^2$$

$$+ \frac{1.3.5}{2^3 3!}(2xt-t^2)^3 + \cdots$$

$$+ \frac{1.3.5.\ldots(2l-1)}{2^l\ l!}(2xt-t^2)^l + \cdots$$

Let us examine the l^{th} term,

$$\frac{1.3.5.\ldots(2l-1)}{2^l\ l!}(2xt-t^2)^l$$

$$= \frac{1.2.3.4.5.\ldots(2l-1)\,2l}{(2.4.6.\ldots 2l)\ 2^l\ l!}(2xt-t^2)^l$$

$$= \frac{(2l)!}{2^l(1.2.3.\ldots.l)\,2^l l!}(2xt-t^2)^l$$

$$= \frac{(2l)!}{2^{2l}(l!)^2}(2xt-t^2)^l$$

Thus, $$G(x,t) = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}(2xt-t^2)^l$$

$$G(x,t) = 1 + \frac{1}{2}(2xt - t^2) + \frac{3}{8}(2xt - t^2)^2 + \frac{15}{48}(2xt - t^2)^3 + \ldots$$

$$= 1 + xt - \frac{1}{2}t^2 + \frac{3}{2}x^2t^2 - \frac{2}{3}xt^3 + \frac{3}{8}t^4 + \frac{5}{2}x^3t^3 - \frac{15}{4}x^2t^4 + \ldots$$

$$= 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \frac{1}{2}(5x^3 - 3x)t^3 + \ldots$$

if $G(x,t) = \sum_{l=0}^{\infty} P_l(x)\, t^l$

then $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

and so on ..

Rodrigues Formula

The polynomials can also be found out by a simple formula. The Rodrigues formula for Legendre Polynomials is

$$P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l}\left((x^2 - 1)^n\right)$$

i.e. $$P_0(x) = \frac{1}{2^0\, 0!}\,(x^2-1)^0 = 1$$

$$P_1(x) = \frac{1}{2^1\, 1!}\,\frac{d}{dx}\left\{(x^2-1)^1\right\} = \frac{1}{2}\cdot 2x = x$$

$$P_2(x) = \frac{1}{2^2\, 2!}\cdot\frac{d^2}{dx^2}\left\{(x^2-1)^2\right\}$$

$$= \frac{1}{4.2}\cdot\frac{d^2}{dx^2}\left(x^4-2x^2+1\right)$$

$$= \frac{1}{8}\left(12x^2-4\right)$$

$$= \frac{1}{2}\left(3x^2-1\right)$$

and so on ....

Question : Prove the recurence relation for Legendre Polynomials -

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

To do this we can start with the generating function for Legendre Polynomials.

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} P_m(x)t^m$$

Differentiating w.r.t t on both sides, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{m=0}^{\infty} mP_m(x)t^{m-1}$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-3/2} = \sum_{m=0}^{\infty} mP_m(x)t^{m-1}$$

Multiplying by $(1-2xt+t^2)$ on both sides we get.

$$(x-t)(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty}(1-2xt+t^2)mP_m(x)t^{m-1}$$

$$\Rightarrow (x-t)\sum_{m=0}^{\infty}P_m(x)t^m = \sum_{m=0}^{\infty}(1-2xt+t^2)mP_m(x)t^{m-1}$$

$$\Rightarrow x\sum_{m=0}^{\infty}P_m(x)t^m - \sum_{m=0}^{\infty}P_m(x)t^{m+1} = \sum_{m=0}^{\infty}mP_m(x)t^{m-1}$$

$$-\sum_{m=0}^{\infty}2xmP_m(x)t^m + \sum_{m=0}^{\infty}mP_m(x)t^{m+1}$$

Equating the co-efficients of t^m on both sides -

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow (n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

Question :- Prove the following recurrence relation :

$$xP'_n - P'_{n-1} = nP_n$$

The recurence relation is -

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_m(x) t^m$$

Differentiating w.r.t. t,

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2} \cdot (-2x+2t) = \sum_{m=0}^{\infty} m P_m(x) t^{m-1} \rightarrow (1)$$

But differentiating the recurrence relation w.r.t. x.

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2} \cdot (-2t) = \sum_{m=0}^{\infty} P'_m(x) t^m \rightarrow (2)$$

Dividing (1) by (2).

$$\frac{x-t}{t} = \frac{\sum_{m=0}^{\infty} m P_m(x) t^{m-1}}{\sum_{m=0}^{\infty} P'_m(x) t^m}$$

$$\Rightarrow (x-t) \sum_{m=0}^{\infty} P'_m(x) t^m = t \sum_{m=0}^{\infty} m P_m(x) t^{m-1}$$

$$\Rightarrow \sum_{m=0}^{\infty} x P'_m(x) t^m - \sum_{n=0}^{\infty} P'_m t^{m+1} = \sum_{m=0}^{\infty} m P_m(x) t^m$$

Equating the coefficients of t^m on both sides.

$$x P'_n(x) - P'_{n-1} = n P_n(x).$$

Question : Prove the orthogonality relation for Legendre Polynomials.

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

The Legendre Differential Equation reads -

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$$

Since the Legendre Polynomials satisfy the equation we can write -

$$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + n(n+1)P_n(x) = 0 \rightarrow (1)$$

and $$(1-x^2)\frac{d^2}{dx^2}P_m(x) - 2x\frac{d}{dx}P_m(x) + m(m+1)P_m(x) = 0 \rightarrow (2)$$

Multiplying (1) by $P_m(x)$ and (2) by $P_n(x)$ and then substracting,

$$(1-x^2)\left[P_m(x)P_n''(x) - P_n(x)P_m''(x)\right]$$
$$-2x\left[P_m(x)P_n'(x) - P_n(x)P_m'(x)\right]$$
$$+\left[n(n+1) - m(m+1)\right]P_m(x)P_n(x) = 0$$

This may be rewritten as-

$$\frac{d}{dx}\left[(1-x^2)\left(P_m(x)P_n'(x) - P_n(x)P_m'(x)\right)\right]$$
$$= \left[m(m+1) - n(n+1)\right]P_m(x)P_n(x)$$

For the Legendre Differential Equation, the -1 and $+1$ are singular points.

So, we shall integrate the above within the limits -1 and $+1$.

$$\int_{-1}^{+1} \frac{d}{dx}\left[(1-x^2)\left(P_m(x)P_n'(x) - P_n(x)P_m'(x)\right)\right] dx$$

$$= \left[m(m+1) - n(n+1)\right]\int_{-1}^{+1} P_m(x)\,P_n(x)\,d$$

$$\Rightarrow \left[(1-x^2)\left(P_m(x)P_n'(x) - P_n(x)P_m'(x)\right)\right]_{-1}^{+1}$$

$$= \left[m(m+1) - n(n+1)\right]\int_{-1}^{+} P_m(x)\,P_n(x)\,dx$$

at $+1$ and -1, $1-x^2 = 0$.

So, $\left[m(m+1) - n(n+1)\right]\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = 0.$

for $m \neq n$, $m(m+1) - n(n+1) \neq 0$

$$\therefore \int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = 0$$

For $m = n$ we have to proceed differently.

Starting with the generating function,

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$$(1-2xt+t^2)^{-1} = \left(\sum_{n=0}^{\infty} P_n(x)t^n\right)\left(\sum_{m=0}^{\infty} P_m(x)t^m\right)$$

Integrating both sides within -1 to $+1$,

$$\int_{-1}^{+1} \frac{1}{1-2xt+t^2}\,dx = \int_{-1}^{+1}\left(\sum_{n=0}^{\infty} P_n(x)t^n\right)\left(\sum_{m=0}^{\infty} P_m(x)t^m\right)$$

In the RHS, for all $m \neq n$ the terms will not survive

So,
$$\int_{-1}^{+1} \frac{1}{1-2xt+t^2}\,dx = \sum_{n=0}^{\infty}\left\{\int_{-1}^{+1} P_n^2(x)\,dx\right\} t^{2n}$$

We now need to evaluate the LHS

Let $z = 1-2xt+t^2$

$\Rightarrow dz = -2t\,dx$

$\Rightarrow dx = -\frac{1}{2t}\,dz$.

Thus. $$\int_{-1}^{+1} \frac{1}{1-2xt+t^2}\,dx = \int_{-1}^{+1} \frac{1}{z}\cdot\left(-\frac{1}{2t}\right)dz$$

$$= -\frac{1}{2t} \ln(z) \Big|_{-1}^{+1}$$

$$= -\frac{1}{2t} \ln\left(1-2xt+t^2\right) \Big|_{-1}^{+1}$$

$$= -\frac{1}{2t} \left[\ln\left(1-2t+t^2\right) - \ln\left(1+2t+t^2\right)\right]$$

$$= -\frac{1}{2t} \left[\ln(1-t)^2 - \ln(1+t)^2\right]$$

$$= -\frac{1}{2t} \left[\ln\left(\frac{1-t}{1+t}\right)^2\right]$$

$$= -\frac{1}{2t} \left[2\ln\left(\frac{1-t}{1+t}\right)\right]$$

$$= -\frac{1}{t} \left[\ln\left(\frac{1-t}{1+t}\right)\right]$$

$$= -\frac{1}{t} \left[\ln(1-t) - \ln(1+t)\right]$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} -$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

So, when expanded the above expression becomes –

$$-\frac{1}{t}\left[\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots\right)-\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-\cdots\right)\right]$$

$$=-\frac{1}{t}\left[-2t-\frac{2t^3}{3}-\frac{2t^5}{5}+\cdots\right]$$

$$=2\left[1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots+\frac{t^{2l}}{2l+1}+\cdots\right]$$

$$=\sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}$$

Thus, $$\sum_{n=0}^{\infty}\left\{\int_{-1}^{+1}P_n^2(x)\,dx\right\}t^{2n}=\int_{-1}^{+1}\frac{1}{1-2xt+t^2}\,dx$$

$$=\sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}$$

Equating the coefficients of t^{2n} on both sides we can say that,

$$\int_{-1}^{+1}P_n(x)P_n(x)\,dx=\frac{2}{2n+1}$$

Hermite Differential Equation.

$$y'' - 2xy' + 2\alpha y = 0$$

We shall employ the Forbenius method to attack this ODE.

We assume a solution of the form:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

Hence, $\quad y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$

and $\quad y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$

Substituting the above in our equation,

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2x \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$+ 2\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2 \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda}$$

$$+ 2\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

Equating the coefficients of the lowest power of x for $\lambda = 0$

$$a_0 \, k(k-1) = 0$$

but $a_0 \neq 0$

Hence, $k = 0, 1$

For $k = 0$;

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda(\lambda-1) x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda (2\alpha - 2\lambda) x^\lambda = 0$$

Equating the coefficients of the general power of x,

$$a_{j+2} (j+2)(j+1) + a_j \, 2(\alpha - j) = 0$$

$$\Rightarrow a_{j+2} = -\frac{2(\alpha - j)}{(j+2)(j+1)} a_j$$

For $j = 0$; $a_2 = -\dfrac{2\alpha}{2.1} a_0$

For $j = 2$; $a_4 = -\dfrac{2(\alpha-2)}{4.3} a_2 = +\dfrac{2^2\alpha(\alpha-2)}{4.3.2.1} a_0$

For $j = 4$; $a_6 = -\dfrac{2(\alpha-4)}{6.5} a_4 = -\dfrac{2^3\alpha(\alpha-2)(\alpha-4)}{6.5.4.3.2.1} a_0$

Generalising, $a_{2n} = (-2)^n \dfrac{\alpha(\alpha-2)(\alpha-4)\ldots(\alpha-(2n-2))}{(2n)!} a_0$

Solution, $y_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

$$= a_0 + a_2 x^2 + a_4 x^4 + \ldots$$

$$= a_0 \left(1 - \frac{2\alpha}{2!} x^2 + \frac{2^2 \alpha(\alpha-2)}{4!} x^4 - \frac{2^3 \alpha(\alpha-2)(\alpha-4)}{6!} x^6 + \ldots\right)$$

For $k = 1$;

$$\sum_{\lambda=0}^{\infty} a_\lambda (\lambda+1)\lambda x^{\lambda-1} + \sum_{\lambda=0}^{\infty} a_\lambda \cdot 2(\alpha - 1 - \lambda) x^{1+\lambda} = 0$$

Equating the coefficients of the general power of x,

$$a_{j+2}(j+3)(j+2) + a_j \cdot 2(\alpha - 1 - j) = 0$$

$$\Rightarrow \quad a_{j+2} = -\frac{2(\alpha-1-j)}{(j+2)(j+3)} a_j$$

For $j=0$; $a_2 = -\frac{2(\alpha-1)}{2.3} a_0$

$j=2$; $a_4 = -\frac{2(\alpha-3)}{4.5} a_2 = +\frac{2^2(\alpha-1)(\alpha-3)}{2.3.4.5} a_0$

$j=4$; $a_6 = -\frac{2(\alpha-5)}{6.7} = -\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{2.3.4.5.6.7} a_0$

Generalising, $a_{2n} = (-2)^n \frac{(\alpha-1)(\alpha-3)\dots(\alpha-(2n-1))}{(2n+1)!} a_0$

Hence, the solution may be constructed as

$$y_2 = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \dots$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \dots$$

$$= a_0 \left(x - \frac{2(\alpha-1)}{3!} x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!} x^5 - \dots \right)$$

Thus, the most general solution can be constructed as

$y = Ay_1 + By_2$

$$= A\left(1 - \frac{2\alpha}{2!}x^2 + \frac{2^2\alpha(\alpha-2)}{4!}x^4 - \frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6 + \dots\right)$$

$$B\left(x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 - \dots\right)$$

Let's say $H_\alpha(x)$ satisfies the Hermite equation.

For even α, we should take y_1 into consideration and for odd α we should take y_2.

Adjusting A and B to preserve the orthogonality condition which we shall learn about later,

We can show that $H_\alpha(x)$ satisfies the Hermite differential equation.

$H_\alpha(x)$ are called the Hermite Polynomials.

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

$H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

$\cdots\cdots$

Question : Write the generating function for Hermite polynomial $H_n(x)$ and show that the generating function leads to the expression

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

The generating function for Hermite polynomials is -

$$G(x,t) = e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Now we are going to derive the expression for $H_n(x)$.

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

To find the coefficient of t^n in the power series expansion of $G(x,t)$, we can differentiate $G(x,t)$ w.r.t. t n-times successively and then evaluate at $t=0$.

Now, $$\frac{d^n}{dt^n}\left(e^{2xt - t^2}\right) = \frac{d^n}{dt^n}\left(\sum_{n=0}^{\infty} \frac{t^m}{m!} H_m(x)\right)$$

$$\Rightarrow \frac{d^n}{dt^n}\left(e^{x^2 - (x^2 - 2xt + t^2)}\right) = \sum_{n=m}^{\infty} \frac{m(m-1)(m-2)\ldots(m-(n-1))}{m(m-1)\ldots(m-(n-1))(m-n)!} H_m(x)$$

$$\Rightarrow e^{x^2} \frac{d^n}{dt^n}\left(e^{-(x-t)^2}\right) = \sum_{n=m}^{\infty} \frac{H_m(x)}{(m-n)!} t^{m-n}$$

On setting $t=0$, all terms in RHS for which $m>n$ will vanish and only the term for $m=n$ will survive.

Also notice that.

$$\frac{d^n}{dt^n}\left(e^{-(x-t)^2}\right)=(-1)^n\frac{d^n}{dx^n}\left(e^{-(x-t)^2}\right).$$

On setting $t=0$ we can therefore write.

$$(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) = H_n(x)$$

Question : A very useful recurrence relation satisfied by Hermite Polynomials is

$$H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x).$$ Prove it.

We can start with the generating fuction for H_n.

$$e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$$

Differentiating both sides w.r.t. t,

$$e^{2xt-t^2}(2x-2t) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}\,n\,t^{n-1}$$

Substituting e^{2xt-t^2} in LHS by $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

$$(2x - 2t) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n t^{n-1}$$

$$\Rightarrow 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n - 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} = \sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

Comparing the coefficients of t on both sides.

$$2x \frac{H_m(x)}{m!} - 2 \frac{H_{m-1}(x)}{(m-1)!} = \frac{H_{m+1}(x)}{m!}$$

$$\Rightarrow 2x \frac{H_m(x)}{m!} - 2m \frac{H_{m-1}(x)}{m!} = \frac{H_{m+1}(x)}{m!}$$

$$\Rightarrow H_{m+1}(x) = 2x H_m(x) - 2m H_{m-1}(x).$$

Question : Prove the following relation for Hermite polynomials.

$$H'_n(x) = 2n H_{n-1}(x).$$

$$\text{or } \frac{d}{dx}\left(H_n(x)\right) = 2n H_{n-1}(x).$$

We can start with the generating function for H_n.

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

Differentiating w.r.t. x,

$$2t\, e^{2xt-t^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n\right)$$

$$\Rightarrow 2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n$$

$$\Rightarrow 2\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n$$

Equating the co-efficients of t^n,

$$2\frac{H_{n-1}(x)}{(n-1)!} = \frac{H_n'(x)}{n!}$$

$$\Rightarrow 2n\frac{H_{n-1}(x)}{n!} = \frac{H_n'(x)}{n!} \Rightarrow H_n'(x) = 2n\, H_{n-1}(x).$$

Question : Prove the mutual orthogonality relation for Hermite polynomials.

The Hermite polynomials $H_n(x)$ are orthogonal over the interval $(-\infty, \infty)$ with respect to the weight function e^{-x^2}.

$$\text{i.e.} \quad \int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) = 0 \quad \text{if } m \neq n$$

To prove this we start with the Hermite differential eq^n.

$$\underline{\underline{H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0.}}$$

Note that,

$$\frac{d}{dx}\left(e^{-x^2} H_n'(x)\right) = \left(e^{-x^2} H_n''(x) + H'(x) \cdot e^{-x^2}(-2x)\right)$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right) = e^{-x^2}\left(H''_n(x) - 2x H'(x)\right)$$

$$\Rightarrow H''_n(x) - 2x H'(x) = e^{x^2} \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right)$$

So, the Hermite differential eq^n can be written as -

$$e^{x^2} \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right) + 2n H_n(x) = 0 \longrightarrow (1)$$

Changing the index to m.

$$e^{x^2} \frac{d}{dx}\left(e^{-x^2} H'_m(x)\right) + 2m H_m(x) = 0$$

Multiplying eq^n (1) by $H_m(x)$ and eq^n (2) by $H_n(x)$ and then subtracting:

$$2(m-n)\, H_m(x) H_n(x) = e^{x^2} H_m(x) \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right)$$

$$- e^{x^2} H_n(x) \frac{d}{dx}\left(e^{-x^2} H_m'(x)\right)$$

$$\Rightarrow 2(m-n)\, e^{-x^2} H_m(x) H_n(x) = H_m(x) \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right) -$$

$$H_n(x) \frac{d}{dx}\left(e^{-x^2} H_m'(x)\right)$$

Integrating from $-\infty$ to $+\infty$.

$$2(m-n) \int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) = \int_{-\infty}^{+\infty} \left(H_m(x) \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right) - H_n(x) \frac{d}{dx}\left(e^{-x^2} H_m'(x)\right)\right) dx$$

$$= \int_{-\infty}^{+\infty} \left(H_m(x) \frac{d}{dx}\left(e^{-x^2} H_n'(x)\right) dx - \int_{-\infty}^{+\infty} H_n(x) \frac{d}{dx}\left(e^{-x^2} H_m'(x)\right)\right) dx$$

$$= 0$$

Thus, if $m \neq n$, $\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) = 0$

Question: Prove the $H_n(-x) = (-1)^n H_n(x)$

The generating function for Hermite polynomials is.

$$G(x,t) = e^{2xt - t^2} = \sum_{n=0}^{\infty} H_m(x) \frac{t^m}{m!}$$

Differentiating w.r.t. t n-times successively.

$$\frac{d^n}{dt^n}\left(e^{2xt-t^2}\right) = \frac{d^n}{dt^n}\left(\sum_{n=0}^{\infty} \frac{t^m}{m!} H_m(x)\right)$$

$$\Rightarrow \frac{d^n}{dt^n}\left(e^{x^2-(x^2-2xt+t^2)}\right) = \sum_{n=m}^{\infty} \frac{m(m-1)(m-2)\dots(m-(n-1))}{m(m-1)\dots(m-(n-1))(m-n)!} H_m(x)$$

$$\Rightarrow e^{x^2}\frac{d^n}{dt^n}\left(e^{-(x-t)^2}\right) = \sum_{n=m}^{\infty} \frac{H_m(x)}{(m-n)!} t^{m-n} \longrightarrow (1)$$

Transforming x to $-x$ we can write,

$$e^{x^2}\frac{d^n}{dt^n}\left(e^{-(-x-t)^2}\right) = \sum_{n=m}^{\infty} \frac{H_m(-x)}{(m-n)!} t^{m-n} \longrightarrow (2)$$

On setting $t=0$, all terms in RHS for which $m > n$ will vanish and only the term for $m=n$ will survive.

Also, $\frac{d^n}{dt^n}\left(e^{-(x-t)^2}\right) = (-1)^n \frac{d^n}{dx^n}\left(e^{-(x-t)^2}\right)$.

But. $$\frac{d^n}{dt^n}\left(e^{-(-x-t)^2}\right) = \frac{d^n}{dt^n}\left(e^{-(x+t)^2}\right)$$

$$= \frac{d^n}{dx^n}\left(e^{-(x+t)^2}\right)$$

On setting $t=0$ we can therefore write.

$$(-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) = H_n(x)$$

and $$(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = H_n(-x)$$

This allows us to write.

$$H_n(-x) = (-1)^n H_n(x)$$

Laguerre's Differential Equation

$$xy'' + (1-x)y' + \alpha y = 0$$

To solve this differential equation we shall employ the Forbenius method.

Assuming a solution of the form:

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \quad , a_0 \neq 0$$

we find that $y' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$

and $y'' = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$

Substituting in the equation,

$$x \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + (1-x) \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$$

$$+ \alpha \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)\{(k+\lambda-1)+1\} x^{k+\lambda-1} + \sum_{\lambda=0}^{\infty} a_\lambda \{\alpha - (k+\lambda)\} x^{k+\lambda} = 0$$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)^2 x^{k+\lambda-1} + \sum_{\lambda=0}^{\infty} a_\lambda \{\alpha - (k+\lambda)\} x^{k+\lambda} = 0$$

The indicial equation reads :-

$$a_0 (k+0)^2 = 0$$

or $k = 0$ (since $a_0 \neq 0$)

For $k=0$,

$$\sum_{\lambda=0}^{\infty} a_\lambda \lambda^2 x^{\lambda-1} + \sum_{\lambda=0}^{\infty} a_\lambda (\alpha-\lambda) x^\lambda = 0$$

Equating the coefficients of the general power of x,

$$a_{j+1} (j+1)^2 x^{j+1-1} + a_j (\alpha - j) x^j = 0$$

$$\Rightarrow \quad a_{j+1} = - \frac{\alpha - j}{(j+1)^2} a_j$$

For $j = 0$, $a_1 = - \dfrac{\alpha}{1^2} a_0$

$j = 1$, $a_2 = - \dfrac{(\alpha-1)}{2^2} a_1 = + \dfrac{\alpha(\alpha-1)}{1^2 . 2^2} a_0$

$j = 2$, $a_3 = - \dfrac{(\alpha-2)}{3^2} a_2 = - \dfrac{\alpha(\alpha-1)(\alpha-2)}{1^2 . 2^2 . 3^2} a_0$

. . .

$$a_n = (-1)^n \frac{\alpha(\alpha-1)(\alpha-2) \dots (\alpha-(n-1))}{(n!)^2}$$

Thus, the solution is -

$$y = a_0 + a_1 x + a_2 x + a_3 x + \dots$$

$$= a_0 \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{m=0}^{n-1} (\alpha - m)}{n!} x^n \right)$$

Similar to the Hermite polynomials we can also obtain Laguerre polynomials for different values of x.

$$L_0(x) = 1$$

$$L_1(x) = -x + 1$$

$$2!\, L_2(x) = x^2 - 4x + 2$$

$$3!\, L_3(x) = -x^3 + 9x^2 - 18x + 6$$

$$4!\, L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$$

$$\dots$$

Generating function : $G(x,h) = \dfrac{e^{-\frac{xh}{1-h}}}{1-h} = \sum_{n=0}^{\infty} L_n(x) h^n$

Rodrigues formula : $L_n(x) = \dfrac{e^x}{n!} \dfrac{d^n}{dx^n} (x^n e^{-x})$