DIAGONALIZATION OF MATRIX A.

$$A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

Step I : Find the eigenvalues using the characteristic eqⁿ.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -3 & -3 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[(-2-\lambda)(2-\lambda) - 1(-3)\right]$$
$$- (-3)\left[3(2-\lambda) - (-1)(-3)\right]$$
$$+ (-3)\left[3 - (-1)(-2-\lambda)\right] = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4 + 3) + 3(3 - 3\lambda) - 3(1-\lambda) = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 1) - 9(\lambda - 1) + 3(\lambda - 1) = 0$$

$$\Rightarrow (\lambda - 1)\left((4-\lambda)(\lambda+1) - 9 + 3\right) = 0$$

$$\Rightarrow (\lambda - 1)(4\lambda - \lambda^2 + 4 - \lambda - 6) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-3\lambda+2)=0$$

$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-2)=0$$

Thus, the eigenvalues are $1, 1, 2$.

Step II : Find the eigenvectors corresponding to the three eigenvalues.

For $\lambda = 1$,

$$(A - I\lambda)x = 0$$

$$\Rightarrow \begin{bmatrix} 4-1 & -3 & -3 \\ 3 & -2-1 & -3 \\ -1 & 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3 & -3 & -3 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow 3x - 3y - 3z = 0$$

$$\Rightarrow x - y - z = 0$$

$$\Rightarrow x = y + z.$$

Two eigenvectors corresponding to $\lambda = 1$ are.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 2$

$$(A - I\lambda)X = 0$$

$$\Rightarrow \begin{bmatrix} 4-2 & -3 & -3 \\ 3 & -2-2 & -3 \\ -1 & 1 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & -3 \\ 3 & -4 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$2x - 3y - 3z = 0 \rightarrow (i)$$

$$3x - 4y - 3z = 0 \rightarrow (ii)$$

$$-x + y = 0 \rightarrow (iii)$$

$$\Rightarrow x = y.$$

Putting $x = y$ in (ii)

$$3x - 4x - 3z = 0$$

$$\Rightarrow -x - 3z = 0$$

$$\Rightarrow x = -3z$$

Eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$

Step III : Adjoin all eigenvectors so as to form a full rank matrix P

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

Step IV : Find P^{-1}

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} = (-1)(-3) = 3$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = (-1)(-1) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -1 - 3 = -4$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (-1)(-1) = 1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3-3 = 0$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = (-1)\,3 = -3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\det P = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 1(3-3) - 0(3-0) + (-1)(1-0)$$

$$= -1.$$

So, $$P^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -4 & 1 \\ 0 & -3 & 1 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Step V : Find the diagonal matrix $D = P^{-1}AP$

$$D = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4+0-3 & 4-3+0 & 12-9+3 \\ 3+0-3 & 3-2+0 & 9-6+3 \\ -1+0+2 & -1+1+0 & -3+3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 6 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1+0 & 1-1+0 & 6-6+0 \\ -3+0+3 & -3+4+0 & -18+24-6 \\ 1+0-1 & 1-1+0 & 6-6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This is the required diagonal matrix.

Note the following :-

1. the diagonal elements of D are the eigenvalues of A.

2. A and D are called "similar matrix" because

   (i) trace of A = trace of D.

   (ii) $\det(A) = \det(D)$

   (iii) eigenvalues of A = eigenvalues of D

Two matrices with equal trace, determinant and eigenvalues are called similar matrices.

the transformation is called similarity transformation.