1 : Prove that the determinant of an orthogonal matrix is either $+1$ or -1.

Suppose, A is an orthogonal matrix,

then $AA^T = I$.

$\Rightarrow \det(AA^T) = \det(I)$

$\Rightarrow \det(A) \cdot \det(A^T) = 1$

Since $\det(A) = \det(A^T)$.

$\therefore \det(A) \cdot \det(A^T) = 1$

$\Rightarrow \{\det(A)\}^2 = 1$

$\Rightarrow \det(A) = \pm 1$

2. Prove that the determinat of an unitary matrix is $1, -1, i$ or $-i$.

Suppose A is an unitary matrix.

then $AA^{\dagger} = I$

or $A(A^T)^* = I$

Now. $\det\left(A(A^T)^*\right) = \det(I)$

$\Rightarrow \det(A)\det\left((A^T)^*\right) = 1$

$\Rightarrow \det(A)\left(\det(A^T)\right)^* = 1$

$\Rightarrow \det(A)\left(\det(A)\right)^* = 1$

$\Rightarrow |\det(A)|^2 = 1$

So, $\det(A)$ must be $+1, -1, +i$ or $-i$

3. Prove that the eigenvalues of a Hermitian matrix are real.

Let us suppose X is an eigenvector corresponding to eigenvalue λ for a Hermitian matrix A.

$$AX = \lambda X$$

$$\Rightarrow X^{\dagger} A X = \lambda X^{\dagger} X \rightarrow (1)$$

$$AX = \lambda X$$

$$\Rightarrow (AX)^{\dagger} = (\lambda X)^{\dagger}$$

$$\Rightarrow X^{\dagger} A^{\dagger} = \lambda^{*} X^{\dagger}$$

$$\Rightarrow X^{\dagger} A^{\dagger} X = \lambda^{*} X^{\dagger} X$$

$$\Rightarrow X^{\dagger} A X = \lambda^{*} X^{\dagger} X \quad \left(\because A = A^{\dagger} \text{ as } A \text{ is hermitian} \right)$$

$$\rightarrow (2)$$

From (1) and (2)

$$\lambda X^{\dagger} X = \lambda^{*} X^{\dagger} X$$

$\underset{a+ib}{\lambda} = \underset{a-ib}{\lambda^{*}}$ holds true only when λ is real.

Hence, eigenvalues of a Hermitian matrix are real.

Q: Prove that the eigenvectors corresponding to different eigenvalues of a Hermitian matrix are orthogonal.

Suppose A is a Hermitian matrix.

λ_i and λ_j are two different eigenvalues.

x_i is an eigenvector corresponding to λ_i

x_j is an eigenvector corresponding to λ_j

$$A x_i = \lambda_i x_i$$

$$\Rightarrow (A x_i)^\dagger = (\lambda_i x_i)^\dagger$$

$$\Rightarrow x_i^\dagger A = \lambda_i^* x_i^\dagger$$

$$\Rightarrow x_i^\dagger A x_j = \lambda_i^* x_i^\dagger x_j$$

$$\Rightarrow x_i^\dagger A x_j = \lambda_i x_i^\dagger x_j \quad \left[\lambda_i = \lambda_i^* \text{ since eigenvalues of Hermitian matrices are real} \right] \rightarrow 1$$

Similarly, $A x_j = \lambda_j x_j$

$$\Rightarrow x_i^\dagger A x_j = \lambda_j x_i^\dagger x_j \rightarrow 2$$

From 1 and 2,

$$\lambda_i x_i^\dagger x_j = \lambda_j x_i^\dagger x_j$$

$$\Rightarrow (\lambda_i - \lambda_j)\, x_i^+ x_j = 0.$$

Since λ_i and λ_j are different,

Hence $\lambda_i - \lambda_j \neq 0$.

So, $x_i^+ x_j$ must be 0.

Similarly we can show that

$$x_j^+ x_i = 0 .$$

This proves that x_i and x_j are orthogonal to each other.

COUNTING UNIQUE ELEMENTS OF MATRICES.

(i) Symmetric Matrix : $A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

For off-diagonal entries.

$$a_{ij} = a_{ji} \quad \text{for} \quad i \neq j$$

Thus, unique off diagonal entries.

$$= \frac{1}{2}\left(\text{total diagonal entries.}\right).$$

Now, total off-diagonal entries = total matrix entries − no. of diagonal entries

$$= n^2 - n$$

$\therefore$ unique off-diagonal entries $= \dfrac{n^2-n}{2}$

For diagonal entries.

there are n unique diagonal entries a_{ii}

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total unique entries = unique off-diagonal entries + unique diagonal entries

$$= \frac{n^2-n}{2} + n = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

(ii) Skew-Symmetric Matrix

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

For off-diagonal entries

$$a_{ij} = -a_{ji} \quad \text{for} \quad i \neq j$$

Thus, unique off diagonal entries.

$= \frac{1}{2}$ (total diagonal entries.).

Now, total off-diagonal entries = total matrix entries − no. of diagonal entries

$= n^2 - n$

$\therefore$ unique off-diagonal entries $= \dfrac{n^2-n}{2}$

For diagonal entries.

there are 0 unique diagonal entries $(a_{ii} = 0)$

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total unique entries = unique off-diagonal entries + unique diagonal entries

$$= \frac{n^2-n}{2} + 0 = \frac{n(n-1)}{2}$$

(iii) Hermitian - matrix $(A_{n\times n})$

$$\begin{bmatrix} a_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} \\ a_{12}-ib_{12} & a_{22} & a_{23}+ib_{23} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_{33} \end{bmatrix}$$

For diagonal entries.

all diagonal entries of a Hermitian - matrix are real.

there are n diagonal entries.

So, there are total n real numbers in n diagonal entries.

For off diagonal entries.

off - diagonal entries of a Hermitian matrix are complex.

there are a total of $n^2 - n$ diagonal entries.

Each entry has two real numbers a_{ij} and b_{ij}.

Hence, there are a total of $2(n^2-n)$ real numbers in off - diagonal entries.

But off-diagonal entries are complex conjugates of each other. Hence, there are

$\frac{2(n^2-n)}{2}$ unique real numbers in off-diagonal entries.

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total unique real numbers $= n + \frac{2(n^2-n)}{2}$

$$= n + n^2 - n$$

$$= n^2$$

(iii) Skew-Hermitian - matrix $(A_{n\times n})$
$$\begin{bmatrix} ib_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} \\ -a_{12}+ib_{12} & ib_{22} & a_{23}+ib_{23} \\ -a_{13}+ib_{13} & -a_{23}+ib_{23} & ib_{33} \end{bmatrix}$$

For diagonal entries.

all diagonal entries of a Hermitian - matrix are purely imaginary.

there are n diagonal entries.

So, there are total n real numbers in n diagonal entries.

For off diagonal entries.

off - diagonal entries of a Hermitian matrix are complex.

there are a total of $n^2 - n$ diagonal entries.

Each entry has two real numbers a_{ij} and b_{ij}.

Hence, there are a total of $2(n^2-n)$ real numbers in off - diagonal entries.

But off-diagonal entries are negative complex conjugates of each other. Hence, there are

$\frac{2(n^2-n)}{2}$ unique real numbers in off-diagonal entries.

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total unique real numbers $= n + \frac{2(n^2-n)}{2}$

$= n + n^2 - n$

$= n^2$