Partial Differential Equations

A partial differential equation is a differential equation that relates a function $u(x_1, x_2, \cdots)$ of two or more variables with the partial derivatives of u w.r.t. those variables.

Some important PDE that we are going to study in this course are linear and second order.

Our objective is to study the following PDE :

(i) $\nabla^2 u = 0$ → Laplace Equation

(ii) $\nabla^2 u = \rho(\vec{r})$ → Poisson Equation

(iii) $\nabla^2 u = \frac{1}{k} \frac{\partial u}{\partial t}$ → Heat Equation / Diffusion Equation

(iv) $\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ → Wave Equation.

To solve and study these equations, we shall employ a technique known as the separation of variables.

The Laplace Equation.

The co-ordinate independent form of the Laplace equation may be written in the following form:

$$\nabla^2 u = 0$$

In electromagnetic theory you shall frequently encounter the Laplace equation with different boundary conditions which shall give you some interesting results.

Let us take the Laplace equation in 2-D cartesian system :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Although other methods of solving this equation do exist, I am going to focus on the technique known as "separation of variables".

We assume that a solution of the form

$$u(x,y) = X(x)\,Y(y) \text{ exist}$$

This yields,

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) = 0$$

$$\Rightarrow Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$$

If the above holds true, then each term must be separately equal to a constant.

i.e. $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = m^2$ and $\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -m^2$

Since X is a function of x and the partial derivative of X is also w.r.t. x,

Hence we may write,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = m^2$$

$$\Rightarrow \frac{d^2 X}{dx^2} - m^2 X = 0$$

The solution to this eqn may be solved using any technique you already know,

$$X(x) = A(y) e^{mx} + B(y) e^{-mx}$$

Similarly, we may write,

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2$$

$$\Rightarrow \frac{d^2Y}{dy^2} + m^2 Y = 0$$

This is our familiar simple harmonic oscillator eq^n. whose general solution is :-

$$Y(y) = C(x)\cos(my) + D(x)\sin(my)$$

Hence we may combine both solutions to obtain the final solution

$$\begin{aligned} u(x,y) &= X(x)\,Y(y) \\ &= \left(A e^{mx} + B e^{-mx}\right)\left(C\cos(my) + D\sin(my)\right) \end{aligned}$$

The coefficients $A(y)$; $B(y)$; $C(x)$ and $D(x)$ may be determined using appropriate boundary conditions.

The Laplace Equation and its solution becomes interesting once we shift to the polar co-ordinates.

We shall try to attack the following forms of the Laplace equation.

1. $\nabla^2 u(s, \phi) = 0$ $\rightarrow$ plane polar co-ordinate system

i.e. $$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial u}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 u}{\partial \phi^2} = 0$$

2. $\nabla^2 u(s, \phi, z) = 0$ $\rightarrow$ cylindrical co-ord. system

i.e. $$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial u}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

3. $\nabla^2 u(r, \theta, \phi) = 0$ $\rightarrow$ spherical polar co-ord system.

i.e. $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) +$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

Case I :

Let's take the simplest case of Laplace equation in the plane polar coordinate system -

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial u}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 u}{\partial \phi^2}=0$$

We shall assume a solution of the type

$$u(s,\phi)=\mathcal{S}(s)\,\Phi(\phi)$$

On substituting, the equation takes the form

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}(\mathcal{S}\Phi)\right)+\frac{1}{s^2}\frac{\partial^2}{\partial \phi^2}\cdot(\mathcal{S}\Phi)=0$$

$$\Rightarrow \frac{\Phi}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \mathcal{S}}{\partial s}\right)+\frac{\mathcal{S}}{s^2}\frac{\partial^2 \Phi}{\partial \phi}=0$$

$$\Rightarrow \frac{\Phi}{s}\left(s\frac{\partial^2 \mathcal{S}}{\partial s^2}+\frac{\partial \mathcal{S}}{\partial s}\cdot\frac{\partial s}{\partial s}\right)+\frac{\mathcal{S}}{s^2}\frac{\partial^2 \Phi}{\partial \phi^2}=0$$

$$\Rightarrow \Phi\frac{\partial^2 \mathcal{S}}{\partial s^2}+\frac{\Phi}{s}\frac{\partial \mathcal{S}}{\partial s}+\frac{\mathcal{S}}{s^2}\frac{\partial^2 \Phi}{\partial \phi^2}=0$$

Dividing by $\mathcal{S}\Phi$ on both sides and then multiplying by s^2

$$\Rightarrow \frac{s^2}{\mathcal{S}}\frac{\partial^2 \mathcal{S}}{\partial s^2} + \frac{s}{\mathcal{S}}\frac{\partial \mathcal{S}}{\partial s} + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

If the above holds true, then the equation may be separated as.

$$\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -n^2 \longrightarrow (1)$$

$$\frac{s^2}{\mathcal{S}}\frac{\partial^2 \mathcal{S}}{\partial s^2} + \frac{s}{\mathcal{S}}\frac{\partial \mathcal{S}}{\partial s} = n^2 \longrightarrow (2)$$

$$(1) \Rightarrow \frac{\partial^2 \Phi}{\partial \phi^2} + n^2 \Phi = 0$$

This is our familiar simple harmonic equation whose solution is -

$$\Phi(\phi) = A(s)\cos(n\phi) + B(s)\sin(n\phi)$$

$$(2) \Rightarrow s^2 \frac{\partial^2 \mathcal{S}}{\partial s^2} + s\frac{\partial \mathcal{S}}{\partial s} - \omega^2 \mathcal{S} = 0$$

The power series solution for this equation is -

$$\mathcal{S}(s) = C(\phi) s^n + D(\phi) s^{-n}$$

Thus we may write the solution as.

$$u(s,\phi) = \left[A\cos(n\phi) + B\sin(n\phi)\right]\left[C s^{n} + D s^{-n}\right]$$

Case II :

We shall now attempt the solution of the Laplace equation in 3-D cylindrical coordinate system:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial u}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

We would love to find the solution of the form

$$u(s,\phi,z) = \mathcal{S}(s)\,\Phi(\phi)\,Z(z)$$

On substituting $u(s,\phi,z)$ in the above equation and dividing by u, we get the following form

$$\frac{1}{\mathcal{S}s}\cdot\frac{\partial}{\partial s}\left(s\frac{\partial \mathcal{S}}{\partial s}\right) + \frac{1}{\Phi s^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z}\cdot\frac{\partial^2 Z}{\partial z^2} = 0$$

The last term depends only on z and the first and second term taken together depend on s and ϕ.

We set, $\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2$

such that it is cast in the following form:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

we can write its solution easily -

$$Z(z) = Ae^{kz} + Be^{-kz}$$

Again, $\frac{1}{S s}\cdot\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{\Phi s^2}\frac{\partial^2 \Phi}{\partial \phi^2} = -k^2$

$$\Rightarrow \frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{\Phi}\cdot\frac{\partial^2 \Phi}{\partial \phi^2} = -k^2 s^2$$

If we set $\frac{1}{\Phi}\cdot\frac{\partial^2 \Phi}{\partial \phi^2} = -m^2$

we can cast it in the form of the SHM equation as follows:

$$\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0$$

The solution is simple -

$$\Phi(\phi) = C\cos(m\phi) + D\sin(m\phi)$$

Now we have the remaining portion :

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+k^2s^2-m^2=0$$

$$\Rightarrow \frac{s}{S}\left(s\frac{d^2S}{ds^2}+\frac{dS}{ds}\right)+\left(k^2s^2-m^2\right)=0$$

$$\Rightarrow s^2\frac{d^2S}{ds^2}+s\frac{dS}{ds}+\left(k^2s^2-m^2\right)S=0$$

This is the Bessel equation whose solution is

$$S(s)=E J_m(ks)+F Y_m(ks).$$

Thus the complete solution takes the form.

$$u(s,\varphi,z)=\left(Ae^{kz}+Be^{-kz}\right)\left(C\cos(m\phi)+D\sin(m\phi)\right)\left(EJ_m(ks)+FY_m(ks)\right)$$

Case III :

Laplace equation in spherical polar co-ordinate system is an important one for a Physicist to learn. Let us try to do it.

The Laplace equation in spherical polar-coordinate system reads,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) +$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

We shall consider that the solution has the form.

$$u(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

Substituting in the equation above -

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(RY)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(RY)\right)$$

$$\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}(RY) = 0$$

$$\Rightarrow \frac{Y}{r^2}\left(r\frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)$$

$$+ \frac{R}{r^2 \sin^2\theta} \cdot \frac{\partial^2 Y}{\partial \phi^2} = 0$$

Multiplying by r^2 and dividing by RB,

$$\frac{1}{R}\left(r \frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r}\right) + \frac{1}{Y \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{Y \sin^2\theta} \cdot \frac{\partial^2 Y}{\partial \phi^2} = 0$$

The first term is dependent on r only and the second and the third terms taken together are dependent on θ and ϕ.

We take $\frac{1}{R}\left(r \frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r}\right) = l(l+1)$

$$\Rightarrow r \frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r} - l(l+1) R = 0$$

This yields the solution,

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

We shall now attack the remaing part of the equation.

$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]=-l(l+1)$$

Multiplying both sides by $Y\sin^2\theta$,

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\phi^2}=-l(l+1)\,Y\sin^2\theta$$

Let's employ the separation of variable technique again to separate out θ and ϕ dependent parts.

Let $\quad Y(\theta,\phi)=A(\theta)\,B(\phi)$

Putting it in the above equation,

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(AB)\right)+\frac{\partial^2}{\partial\phi^2}(AB)=-l(l+1)\,AB\sin^2\theta$$

$$\Rightarrow B\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+A\frac{\partial^2 B}{\partial\phi^2}+l(l+1)\,AB\sin^2\theta=0$$

Dividing both sides by AB,

$$\left[\frac{\sin\theta}{A}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+l(l+1)\sin^2\theta\right]+\left[\frac{1}{B}\frac{\partial^2 B}{\partial\phi^2}\right]=0$$

We see that the first term depends only on θ and the second term depends only on ϕ

So each must be equal to some constant.

Let's say,

$$\frac{\sin\theta}{A}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+l(l+1)\sin^2\theta = m^2$$

So, the other term must be $-m^2$

$$\therefore \frac{1}{B}\frac{\partial^2 B}{\partial\phi^2} = -m^2$$

Since, B depends only on ϕ i.e. $B(\phi)$

$\therefore$ partial derivative may be converted to a total derivative.

$$\frac{\partial^2 B}{\partial\phi^2} \rightarrow \frac{d^2 B}{d\phi^2}$$

$$\frac{1}{B}\frac{d^2B}{d\phi^2} = -m^2$$

$$\Rightarrow \frac{d^2B}{d\phi^2} = -Bm^2$$

$$\Rightarrow \frac{d^2B}{d\phi^2} + Bm^2 = 0$$

The solution to this equation is simple :

$$\boxed{B(\phi) = K_1 e^{im\phi}}$$

'm' can take both positive & negative values.

Let's try to solve the other equation

$$\frac{\sin\theta}{A}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + l(l+1)\sin^2\theta = m^2$$

Since A is a function of θ alone, so we can replace the partial derivative with total derivative

$$\frac{\sin\theta}{A}\frac{d}{d\theta}\left(\sin\theta\frac{dA}{d\theta}\right)+l(l+1)\sin^2\theta = m^2$$

$$\Rightarrow \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dA}{d\theta}\right)+l(l+1)\sin^2\theta A = m^2 A$$

$$\Rightarrow \sin^2\theta\frac{d^2A}{d\theta^2}+\sin\theta\cos\theta\frac{dA}{d\theta}+l(l+1)\sin^2\theta A = m^2 A$$

$$\Rightarrow \frac{d^2A}{d\theta^2}+\cot\theta\frac{dA}{d\theta}+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]A = 0$$

Let, $x = \cos\theta$

$$\Rightarrow \theta = \cos^{-1}x$$

$$\Rightarrow d\theta = \frac{-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow \frac{d}{d\theta} = -\sqrt{1-x^2}\frac{d}{dx}$$

$$\Rightarrow \frac{d^2}{d\theta^2} = -\sqrt{1-x^2}\frac{d}{dx}\left(-\sqrt{1-x^2}\frac{d}{dx}\right)$$

$$= \sqrt{1-x^2}\sqrt{1-x^2}\frac{d^2}{dx^2}+\frac{\sqrt{1-x^2}(-2x)}{2\sqrt{1-x^2}}\frac{d}{dx}$$

$$= (1-x^2)\frac{d^2}{dx^2}-x\frac{d}{dx}$$

$$\frac{d^2A}{d\theta^2} + \cot\theta \frac{dA}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta} \right] A = 0$$

$$\Rightarrow (1-x^2)\frac{d^2A}{dx^2} - x\frac{dA}{dx} + \cot\theta\left(-\sqrt{1-x^2}\right)\frac{dA}{dx}$$

$$+ \left[l(l+1) - \frac{m^2}{1-\cos^2\theta} \right] A = 0$$

$$\Rightarrow (1-x^2)\frac{d^2A}{dx^2} - x\frac{dA}{dx} - \frac{\cos\theta}{\sin\theta}\sqrt{1-\cos^2\theta}\,\frac{dA}{dx}$$

$$+ \left[l(l+1) - \frac{m^2}{1-\cos^2\theta} \right] A = 0$$

$$\Rightarrow (1-x^2)\frac{d^2A}{dx^2} - x\frac{dA}{dx} - \cos\theta\frac{dA}{dx}$$

$$+ \left[l(l+1) - \frac{m^2}{1-x^2} \right] A = 0$$

$$\Rightarrow (1-x^2)\frac{d^2A}{dx^2} - 2x\frac{dA}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2} \right] A = 0$$

This equation is known as the Associated Legendre's differential equation.

Anyways, the solution to the Associated Legendre's differential equation is the associated Legendre function -

$$P_l^m(x) = (1-x^2)^{\frac{|m|}{2}} \left(\frac{d}{dx}\right)^{|m|} P_l(x)$$

$P_l(x)$ is the Legendre Polynomial defined by

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2-1)^l$$

But we considered $\cos\theta = x$.

$\therefore$ The solution that we want is.

$$A(\theta) = P_l^m(\cos\theta) = (1-\cos^2\theta)^{\frac{|m|}{2}} \left(\frac{d}{d\cos\theta}\right)^{|m|} \left\{ \frac{1}{2^l l!} \left(\frac{d}{d\cos\theta}\right)^l (\cos^2\theta - 1)^l \right\}$$

Simply put, the solution of the Laplace equation in 3D spherical polar coordinate system has the form :

$$u(r,\theta,\phi) = \left(Ar^l + \frac{B}{r^{l+1}}\right) . P_l^m(\cos\theta) . e^{im\phi}$$

The Wave Equation

We shall now use the general method of separation of variables for the 3-D cartesian co-ordinate form of the wave equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

We substitute $u(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$ in the above equation and then divide both sides by $X(x)\,Y(y)\,Z(z)\,T(t)$.

This gives.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\cdot\frac{\partial^2 Z}{\partial z^2} = \frac{1}{c^2 T}\cdot\frac{\partial^2 T}{\partial t^2}$$

Let us consider.

$$\frac{1}{c^2 T}\cdot\frac{\partial^2 T}{\partial t^2} = -\mu^2$$

$$\Rightarrow \quad \frac{d^2 T}{dt^2} + \mu^2 c^2 T = 0$$

The solution to this eq^n is given by.

$$T(t) = A_t\, e^{-i\mu ct} + B_t\, e^{i\mu ct}$$

If we consider $\mu^2 = -(l^2 + m^2 + n^2)$

we can write the following equations.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2$$

$$\frac{1}{Z}\cdot\frac{d^2Y}{dz^2} = -n^2$$

In that case, the solutions to the above are.

$$X(x) = A_x e^{-ilx} + B_x e^{ilx}$$

$$Y(y) = A_y e^{-imy} + B_y e^{imy}$$

$$Z(z) = A_z e^{-inz} + B_z e^{inz}$$

We can combine these to obtain the solution to the wave equation,

$$u(x,y,z,t) = A e^{-i(\mu ct + lx + my + nz)}$$

where $A = A_t A_x A_y A_z$.
and $B_t, B_x, B_y, B_z = 0$

In the language of physics, $\mu c = \omega$ (angular frequency)

and. l, m, n are the components of the wave vector $k = \frac{2\pi}{\lambda}$.

ie. $k_x = l$, $k_y = m$, $k_z = n$

This prompts us to write

$$u(x,y,z,t) = Ae^{-i(k_x x + k_y y + k_z z + \omega t)}$$

$$= Ae^{-i(\vec{k}.\vec{r} + \omega t)}$$

Had we taken $A_x, A_y, A_z = 0$

and $A = A_t B_x B_y B_z$

we would have obtained

$$u(x,y,z,t) = A\, e^{-i(\vec{k}.\vec{r} - \omega t)}.$$

Both are equally valid solutions and represents a travelling wave.

The wave equation may be solved in any physical system (and in any co-ordinate systems of our choice) to obtain the properties of the waves generated in the system.

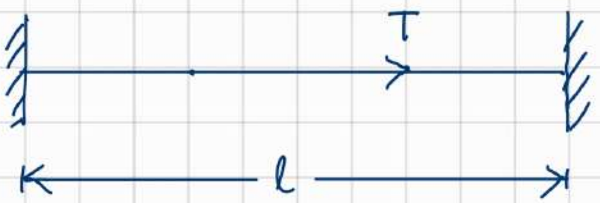
Let us try a simple system :-

1. The wave equation for a stretched spring.

The speed of wave depends on the characteristic of the medium.

For an uniform string, the characteristic that determines the speed is the linear density (mass per unit length), μ.

$$\text{ie.} \quad \mu = \frac{\text{mass of string}}{\text{total length of string}} = \frac{m}{l}$$



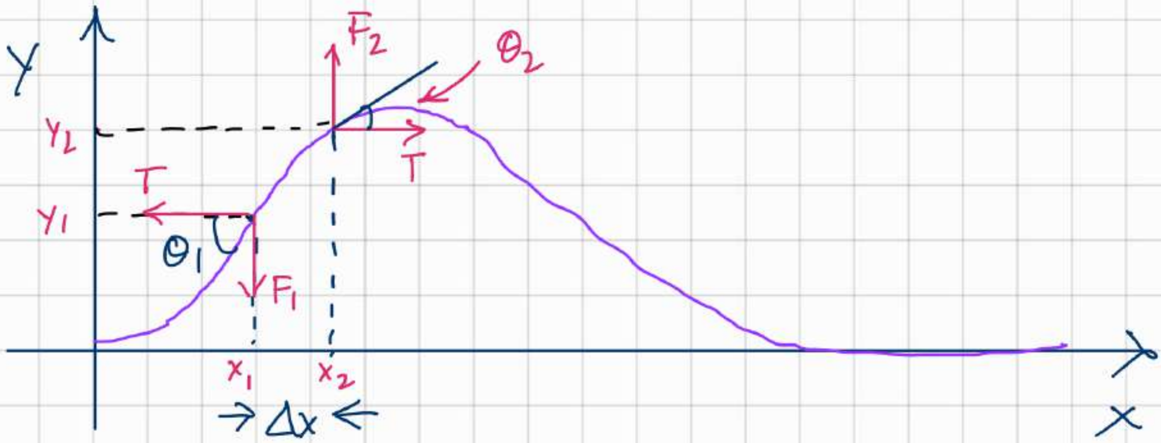
Let us first verify that the wave equation can indeed be applied to this system.

The mass (Δm) of a small length of string (Δx) is

$$\Delta m = \mu \Delta x$$

When the string is in equilibrium under a tension is at rest the tension (T) must be constanant.

If we pluck the string a tranverse wave moves in the positive x-direction. The small mass element oscillates perpendicular to the wave motion as a result of the restoring force provided by the string and does not move in the x-direction.



The net force on the element of the string, acting parallel to the string, is the sum of the tension in the string and the restoring force.

The x-components of the force of tension cancel, so the net force is equal to the sum of the y-components of the force.

The magnitude of the x-component of the force is equal to the horizontal force of tension of the string T as evident in the figure above.

$\tan\theta$ is equal to the partial derivative of y with respect to x.

$$\left.\frac{\partial y}{\partial x}\right)_{x=x_1} \tan\theta_1 = -\frac{F_1}{T} \quad \text{and} \quad \left.\frac{\partial y}{\partial x}\right)_{x=x_2} = \tan\theta_2 = \frac{F_2}{T}$$

The net force on small mass element can be written as

$$F_{net} = F_1 + F_2 = T\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right]$$

Using Newton's second law.

$$F_{net} = \text{mass} \times \text{acceleration}.$$

$$\Rightarrow T\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right] = \Delta m\, a = \mu \Delta x \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right]}{\Delta x} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right]}{\Delta x} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

The wave equation that we know is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Thus, speed, $c = \sqrt{\frac{T}{\mu}}$

Once we obtain this equation, the solution is not at all difficult. Try it yourself!

H/W : Just like the Laplace equation, try to write the Wave Equation in 2-D plane polar coordinates, 2-D cartesian co-ordinate system, 3-D spherical polar coordinate system and 3-D cylindrical system.

Some may represent an actual physical system, some may not.

See if you can obtain a general solution in each case.

The Heat Equation/ Diffusion Equation.

The heat equation or diffusion equation simply reads:

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \quad k \rightarrow \text{diffusivity}$$

$\nabla^2 u$ takes different forms depending on the coordinate system that we use.

HW: write the diffusion equation in cartesian, polar and cylindrical coordinate systems in 2-D and 3-D.

This equation holds a special place in physics as it describes how the temperature or concentration of a substance change over time and space due to heat transfer or diffusion.