

Caley - Hamilton Theorem .

Every square matrix satisfies its own characteristic equation - $|A - \lambda I| = 0$

For eg :- $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

According to the Caley - Hamilton Theorem , the matrix A shall satisfy this equation

$$\text{ie. } A^2 - 4A - 5 = 0$$

Let us verify this .

$$A^2 - 4A - 5I$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Finding A^{-1} using the Cayley-Hamilton Theorem

The characteristic equation is -

$$\lambda^2 - 4\lambda - 5 = 0$$

According to the Cayley-Hamilton Theorem,

$$A^2 - 4A - 5I = 0$$

$$\Rightarrow AAA^{-1} - 4AA^{-1} - 5IA^{-1} = 0$$

$$\Rightarrow A - 4I - 5A^{-1} = 0$$

$$\Rightarrow 5A^{-1} = A - 4I$$

$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5} \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 & 4/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 + 8/5 & 4/5 - 4/5 \\ -6/5 + 6/5 & 2/5 - 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 5/5 & 0 \\ 0 & -1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$