Topics

- basic definition
- addition and multiplication.
- geometrical interpretation
- modulus and complex conjugate
- roots of a complex number

* Complex Numbers.

represented as $z = x + iy$. where $i = \sqrt{-1}$.

real part of z : $Re(z) = x$

imaginary part of z : $Im(z) = y$

—

* Addition and multiplication of two complex numbers.

suppose. $z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

* Comparing two complex numbers.

$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

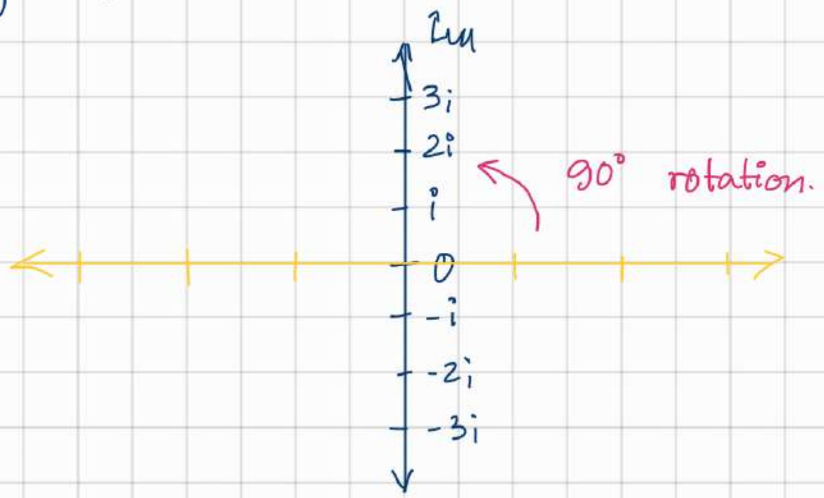
Saying, $z_1 \cancel{>} z_2$ carries no meaning.

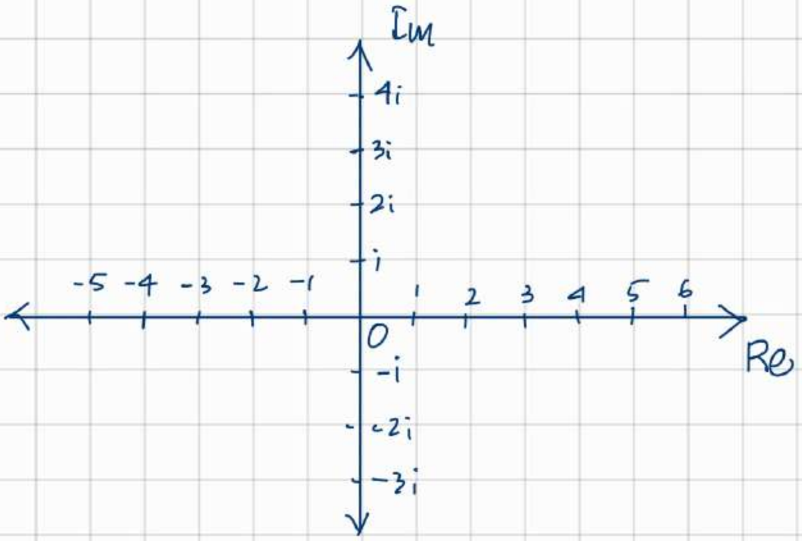
We may say $|z_1| > |z_2|$

* Geometrical Interpretation of Complex Numbers.

Real axis →

Multiplying the real numbers by i gives the imaginary axis.
We can think of "multiplying by i" as rotating the real axis by 90°.

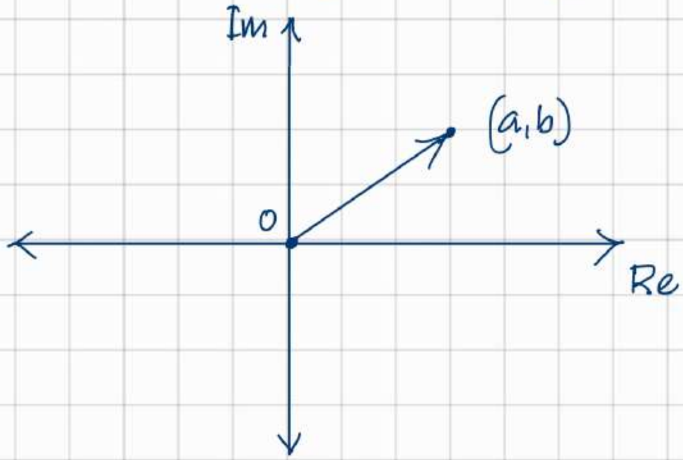




↳ Also called argand plane or complex plane.

Any complex number may be represented on the complex plane as a point

$$(a+ib) \equiv (a,b)$$



Modulus of a complex number

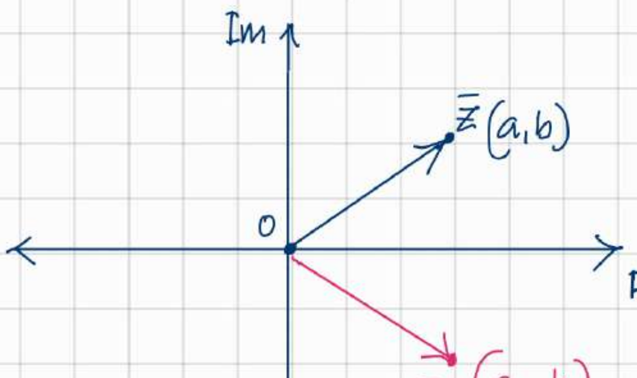
- gives the distance of the complex number from the origin on the Argand plane.

if $\quad z = a+ib$

$$|z| = \sqrt{a^2+b^2}$$

Conjugate of a complex number

- reflection of the complex number about the real axis.

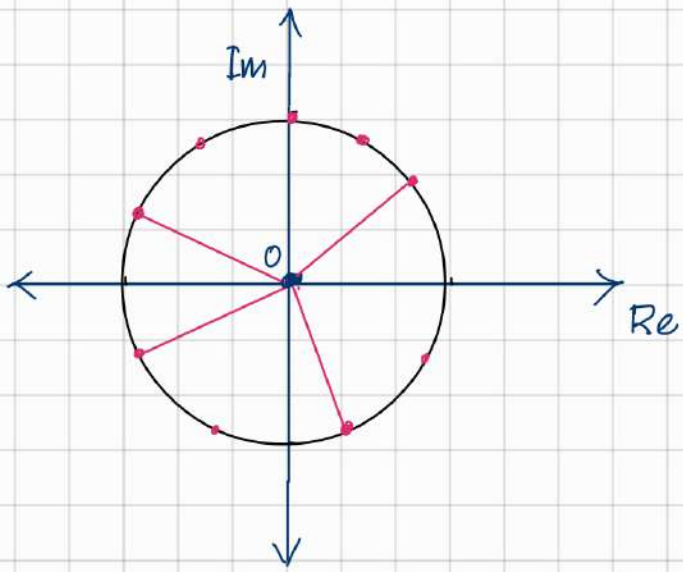


complex conjugate of $\quad z = a+ib \rightarrow \bar{z} = a-ib$

Useful properties or relations.

1) $|z| \geq 0$

2) All complex numbers with the same modulus lie on a circle with its centre at the origin.



3) $|z_1 z_2| = |z_1||z_2|$

Proof: Consider $z_1 = a + ib$ $\quad z_2 = c + id$

H/W Hence, show that $|z_1 z_2| = |(a+ib)(c+id)|$

$= \text{------}$

$= (a^2+b^2)(c^2+d^2)$

$= |z_1||z_2|$

4) $\left| z_1 / z_2 \right| = \dfrac{|z_1|}{|z_2|}$

H/W Try to prove it !

5) Triangle inequality.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

H/W

Since you can represent complex numbers as points on the argand plane, can you think of complex numbers as vectors and try to justify the above relation.

6) $z + \bar{z} = 2\mathrm{Re}(z)$

$z - \bar{z} = 2i\,\mathrm{Im}(z)$

Geometrical interpretation?

7) $z\bar{z} = |z|^2$

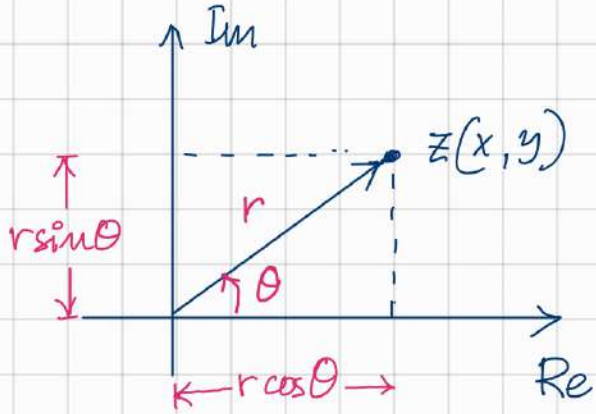
8) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

H/W prove it by considering $z_1 = a+ib$, $z_2 = c+id$

Geometrical interpretation of multiplication of two complex numbers.

To intuitively understand, it is useful to represent the complex no.s in its polar form.



$$z = x + iy$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(x/y\right)$$

or $x = r\cos\theta$; $y = r\sin\theta$

Hence, $z = r\cos\theta + ir\sin\theta$

$$= r\left(\cos\theta + ir\sin\theta\right)$$

We know,

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

Thus, the polar form of $z = x + iy$ is

$$\underline{\underline{z = re^{i\theta}}}$$

↳ most useful notation #

(H/W) If $z = x + iy = re^{i\theta}$ then find the complex conjugate of z in polar form.

i.e. $\bar{z} = x - iy = ?$

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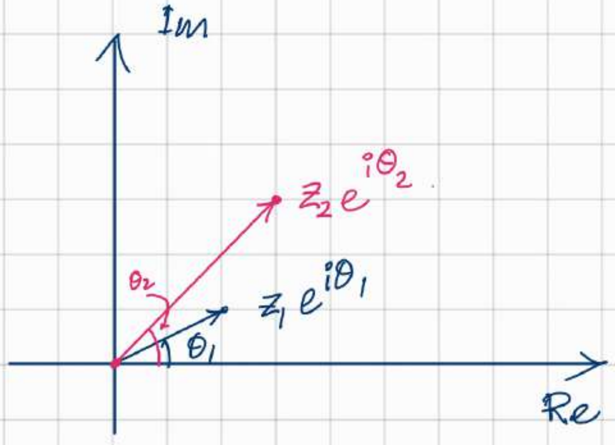
We can now interpret the multiplication of two complex numbers geometrically.

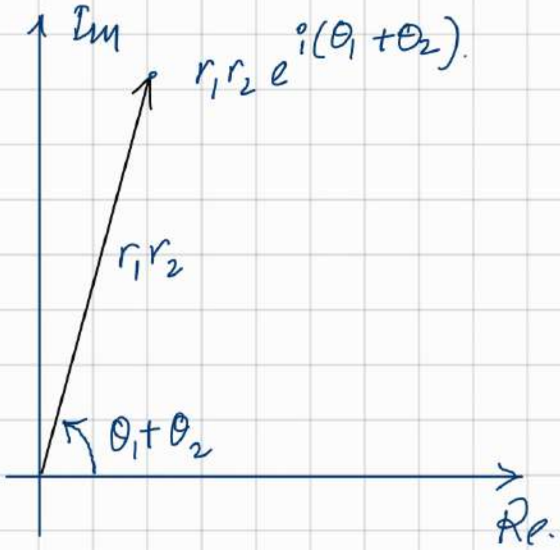
Consider two complex nos.
$z_1 = r_1 e^{i\theta_1}$ and
$z_2 = r_2 e^{i\theta_2}$.



Now. $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$= R e^{i\theta}$

$R = r_1 r_2 \; ; \; \theta = \theta_1 + \theta_2$



Thus, we can think of multiplication as rotation and scaling.

Argument of a complex number

Consider a complex number

$z = re^{i\theta}$ $\rightarrow$ θ is the argument.

We can rotate it by $2n\pi$ and it will still be the same point on the Argand plane.

$$z = re^{i(\theta+2n\pi)}$$
$$= re^{i\theta} \cdot e^{i2n\pi}$$
$$= re^{i\theta}\left\{\cos(2n\pi) + i\sin(2n\pi)\right\}$$

$$\underbrace{\arg(z)}_{\text{general argument}} = \underbrace{\mathrm{Arg}(z)}_{\text{principal argument}} + 2n\pi$$

$$\boxed{-\pi < \mathrm{Arg}(z) \leq \pi}$$

de Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

It is very easy to verify in terms of complex no.s.

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow (re^{i\theta})^n = r^n(\cos\theta + i\sin\theta)^n$$

$$\Rightarrow r^n e^{in\theta} = r^n(\cos\theta + i\sin\theta)^n$$

$$\Rightarrow r^n\{\cos(n\theta) + i\sin(n\theta)\} = r^n(\cos\theta + i\sin\theta)^n$$

$$\Rightarrow \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

Finding roots of a complex number.

Suppose z is a complex number.

We need to find the m^{th} root of z

i.e. $\sqrt[m]{z}$ or $z^{1/m}$

$=$.

For a real number $x = 4$.

$x^{1/2} = \pm 2$.

roots are symmetric about the origin on the real axis.

Do we see something similar in case of complex no.s ?

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Let, $z = re^{i\theta} = re^{i(\theta + 2n\pi)}$

roots, $z_0 = z^{1/m} = r^{\frac{1}{m}} e^{i\left(\frac{\theta}{m} + \frac{2n\pi}{m}\right)}$

If $n = m$, there are infinite solutions.

as. $z_0 = r^{\frac{1}{m}} e^{i\left(\theta/m + 2\pi\right)}$

Hence we apply the restriction, $n = 0, 1, 2, \ldots, (m-1)$

For example if $m = 3$, $n = 0, 1, 2$

In that case the roots will be

$$z_{01} = r^{1/m} e^{i\theta/m} = r^{1/3} e^{i\theta/3}$$

$$z_{02} = r^{1/m} e^{i(\theta/m + \frac{2}{m}\pi)} = r^{1/3} e^{i(\theta/3 + \frac{2}{3}\pi)}$$

$$z_{03} = r^{1/m} e^{i(\theta/m + \frac{4}{m}\pi)} = r^{1/3} e^{i(\theta/3 + \frac{4}{3}\pi)}$$

Example 1 : $z = i$; $z^{1/2} = ?$

$$i = 1\, e^{i\pi/2}$$

$$= e^{i(\pi/2 + 2n\pi)}$$

$$i^{1/2} = e^{i(\pi/4 + n\pi)}$$

Since, $m = 2$, $n = 0, 1.$

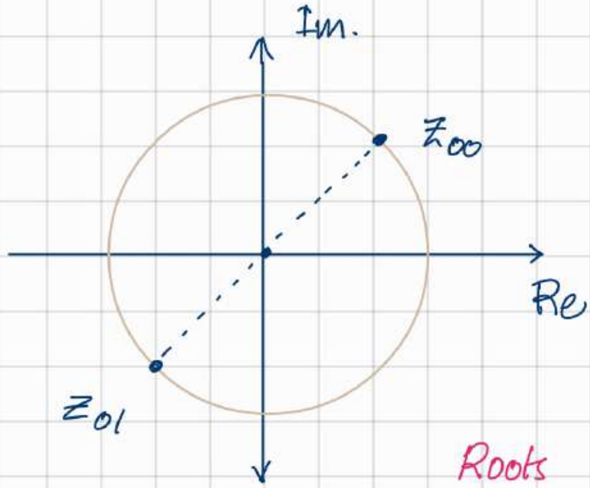
The roots are. $z_{00} = e^{i\pi/4}$

$$z_{01} = e^{i(\pi/4 + \pi)}$$

$$z_{00} = e^{i\pi/4} = \cos \pi/4 + i \sin \pi/4$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_{01} = e^{i(\pi/4+\pi)}$$
$$= \cos\left(\pi/4+\pi\right) + i\sin\left(\pi/4+\pi\right)$$
$$= -\cos\left(\pi/4\right) + i\left(-\sin\pi/4\right)$$
$$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$



Roots appear symmetrically.

Example 2 : $z = -1 \qquad z^{1/3} = ?$

$$z = -1 = 1 \cdot e^{i(\pi+2n\pi)}$$
$$z_0 = z^{1/3} = e^{i\left(\pi/3 + \frac{2n\pi}{3}\right)}$$

$n = 0, 1, 2.$

$$z_{00} = e^{i\pi/3} = \cos\left(\pi/3\right) + i\sin\left(\pi/3\right)$$
$$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

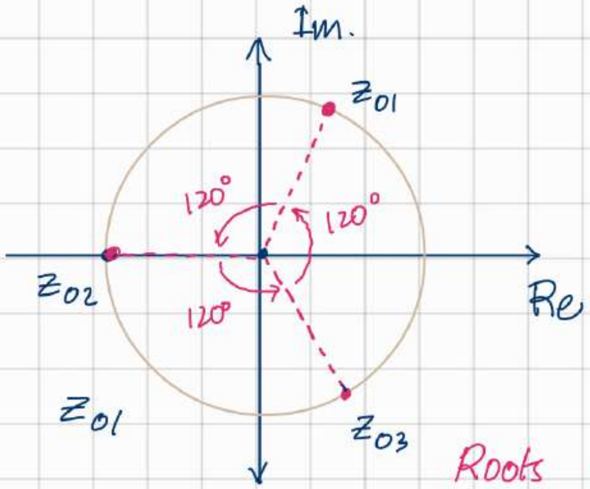
$$z_{01} = e^{i\left(\pi/3 + \frac{2\pi}{3}\right)} = e^{i(3\pi/3)} = e^{i\pi} = -1$$

$$z_{02} = e^{i\left(\pi/3 + \frac{4\pi}{3}\right)} = e^{i(5\pi/3)} = e^{i(2\pi - \pi/3)}$$

$$= \cos(2\pi - \pi/3) + i\sin(2\pi - \pi/3)$$

$$= \cos(\pi/3) + i(-\sin\pi/3)$$

$$= \frac{1}{\sqrt{2}} - i\frac{\sqrt{3}}{2}$$



Roots appear symmetrically.

Example 3 : $z = -1$, $z^{1/4} = ?$

$$z = -1 = 1e^{i(\pi + 2n\pi)}$$

$$z_0 = z^{1/4} = e^{i\left(\pi/4 + \frac{2n\pi}{4}\right)}$$

$n = 0, 1, 2, 3.$

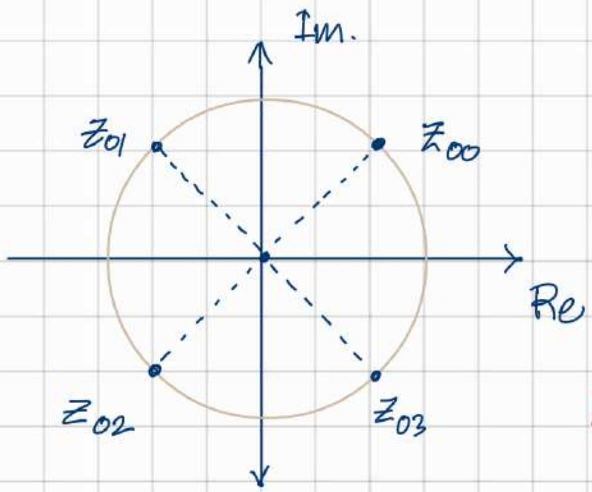
$$z_{00} = e^{i(\pi/4)} = \cos(\pi/4) + i\sin(\pi/4)$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_{01} = e^{i(\pi/4 + \pi/2)} = \cos(\pi/4 + \pi/2) + i\sin(\pi/4 + \pi/2)$$
$$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_{02} = e^{i(\pi/4 + \pi)} = \cos(\pi/4 + \pi) + i\sin(\pi/4 + \pi)$$
$$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$z_{03} = e^{i(\pi/4 + \frac{3}{2}\pi)} = \cos(\pi/4 + 3.\pi/2) + i\sin(\pi/4 + 3\pi/2)$$
$$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$



Roots appear symmetrically.

Homework : Find the square root of $z = (1+i)^{\frac{1}{2}}$

i.e. find $z_0 = z^{1/2} = \sqrt{(1+i)}^{1/2}$

and plot the roots on the Argand plane.