COMPLEX NUMBERS

$\hookrightarrow$ $z = a + ib$

$\hookrightarrow$ $Re(z) = a$; $Im(z) = b$

Complex Conjugate

$$z = a + ib \longrightarrow a - ib.$$

Modulus of z

$$|z| = \sqrt{a^2+b^2}$$

$$|z| = \sqrt{a^2+b^2}$$

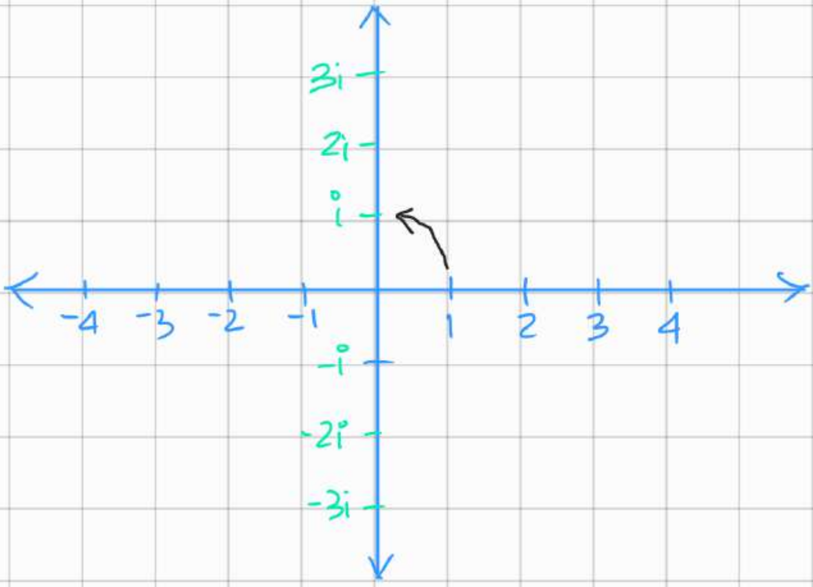
* Inequality

$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$

$z_1 > z_2$ has no meaning

$|z_1| > |z_2|$ has meaning

GEOMETRICAL INTERPRETATION



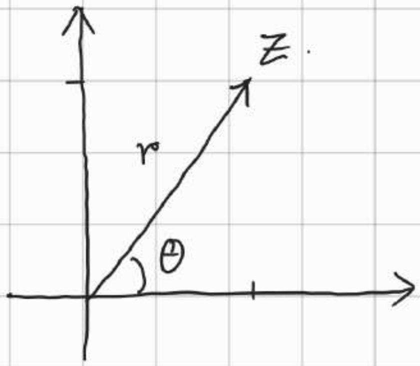
Multiplication by i means a 90° rotation

POLAR FORM

$$z = x + iy$$

Polar form :- $r[\cos\theta + i\sin\theta]$

$$= re^{i\theta}$$



$* \ r = \sqrt{(x^2 + y^2)} = |z|$

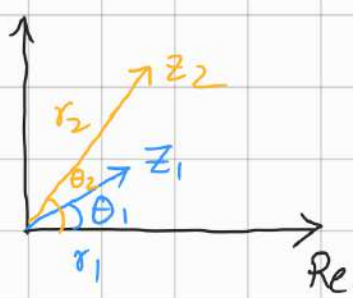
$* \ x = r\cos\theta \quad ; \quad y = r\sin\theta$

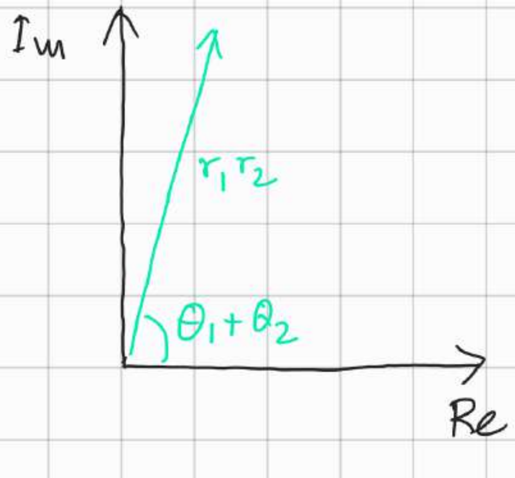
$* \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

What does $z_1 z_2$ mean geometrically ?

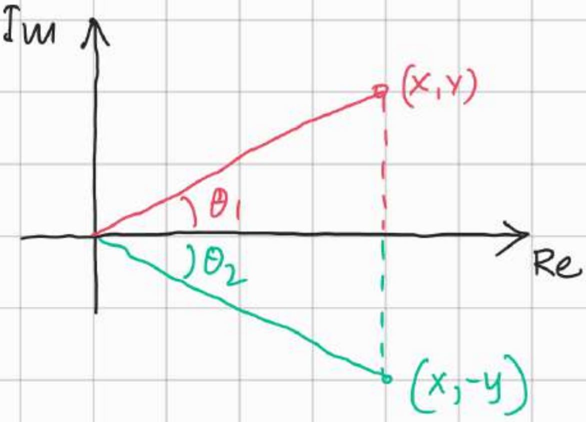
$z_1 = r_1 e^{i\theta_1} \quad ; \quad z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = (r_1 . r_2)\, e^{i(\theta_1+\theta_2)}$





What does complex conjugate mean geometrically?



$$x = \frac{z + \bar{z}}{2}$$

$$y = \frac{z - \bar{z}}{2i}$$

$$z = re^{i\theta} \quad ; \quad \bar{z} = re^{-i\theta}$$

complex conjugate → image w.r.t the real axis

Argument of z

$$re^{i\theta} = re^{i(\theta + 2n\pi)}$$

$$\arg(z) = \underbrace{\mathrm{Arg}(z)}_{\text{Principal Argument}} + 2n\pi$$

$$-\pi < \mathrm{Arg}(z) \leq \pi$$

DE MOIVRE'S THEOREM

$$z = re^{i\theta} = r\left[\cos\theta + i\sin\theta\right]$$

$$z^n = r^n e^{in\theta} = r\left[\cos(n\theta) + i\sin(n\theta)\right]$$

SOME USEFUL NOTATIONS

(i) $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$$i = e^{i\frac{\pi}{2}}$$

(ii) $e^{i(2n\pi)} = \cos(2n\pi) + i\sin(2n\pi) = 1$

$$1 = e^{i(2n\pi)}$$

(iii) $e^{i(2n+1)\pi} = \cos\left[(2n+1)\pi\right] + i\sin\left[(2n+1)\pi\right] = -1$

$$-1 = e^{i(2n+1)\pi}$$

$$\boxed{e^{i\pi} + 1 = 0}$$

ROOTS OF A COMPLEX NO.

$$z = re^{i\theta} = re^{i(\theta + 2n\pi)}$$

We need to find the m^{th} root ie $z^{\frac{1}{m}}$

$$z^{\frac{1}{m}} = r^{\frac{1}{m}} e^{i\left(\frac{\theta + 2n\pi}{m}\right)}$$

$$= \sqrt[m]{r}\, e^{i\left(\frac{\theta}{m} + \frac{2n}{m}\pi\right)}$$

What values can 'n' take ?

$\hookrightarrow n = \underline{1, 2, \ldots\ldots}, (m-1)$

Examples :- $z = i$ find $z^{\frac{1}{2}}$

$$z = i = e^{i\left[2n\pi + \frac{\pi}{2}\right]}$$

$$z^{\frac{1}{2}} = e^{i\left[n\pi + \frac{\pi}{4}\right]}$$

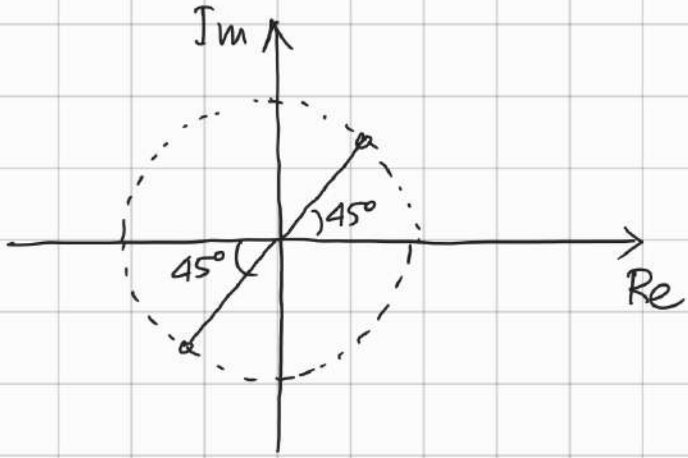
Since, $m = 2$; $n = 0, 1$

$$\therefore \quad i^{\frac{1}{2}} = e^{i\frac{\pi}{4}} \quad \text{and} \quad e^{i\frac{5\pi}{4}}$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \; ; \; \cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \; ; \; -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}.$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \; ; \; -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$



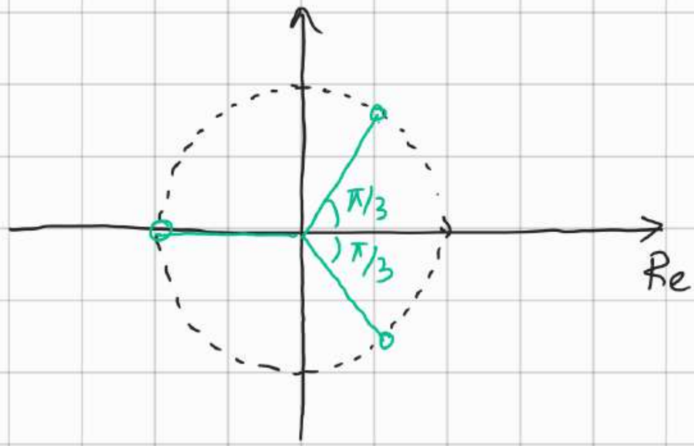
Example :- $z = -1$ find $z^{\frac{1}{3}}$

$$z = e^{i(\pi + 2n\pi)}$$

$$z^{\frac{1}{3}} = e^{i\left(\frac{\pi}{3} + \frac{2n\pi}{3}\right)} \quad ; \text{ Here } m = 3 \ \therefore n = 0, 1, 2$$

$$= e^{i\left(\frac{\pi}{3} + 0\right)} \ ; \ e^{i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} \ ; \ e^{i\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)}$$

$$= e^{i\frac{\pi}{3}} \ ; \ e^{i\pi} \ ; \ e^{i5\frac{\pi}{3}}$$



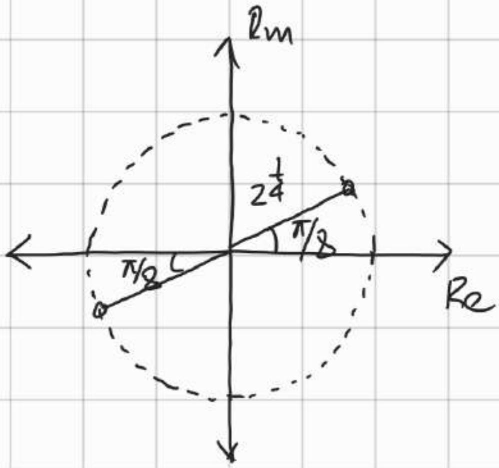
Example :- $z = 1+i$ find $z^{\frac{1}{2}}$

$$z = re^{i\theta}$$
$$= \sqrt{1^2+1^2}\, e^{i\left(\tan^{-1}\frac{1}{1}\right)}$$
$$= \sqrt{2}\, e^{i\pi/4}$$

$$z^{\frac{1}{2}} = 2^{\frac{1}{4}}\, e^{i\left(\frac{\pi}{8} + \frac{2n\pi}{2}\right)}$$
$$= 2^{\frac{1}{4}}\, e^{i\left(\frac{\pi}{8} + n\pi\right)}$$

$m = 2 \quad \therefore \quad n = 0, 1.$

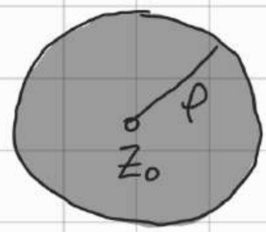
$$\therefore\ z^{\frac{1}{2}} = 2^{\frac{1}{4}}\, e^{i\frac{\pi}{8}} \ ;\ 2^{\frac{1}{4}}\, e^{i\left(\pi + \frac{\pi}{8}\right)}$$



REGIONS IN A COMPLEX PLANE

DISK

$$|z - z_0| \leq \rho$$



included points are called neighbourhood of z_0.

if $0 < |z - z_0| < \rho$

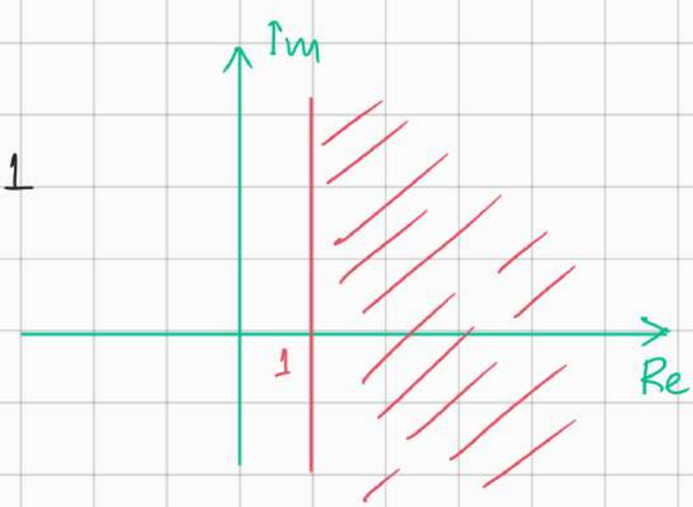
it means all points are taken except 'z_0'

then this disk is called 'punctured disk'.

Examples:

Example 1:-

region : $Re(z) > 1$

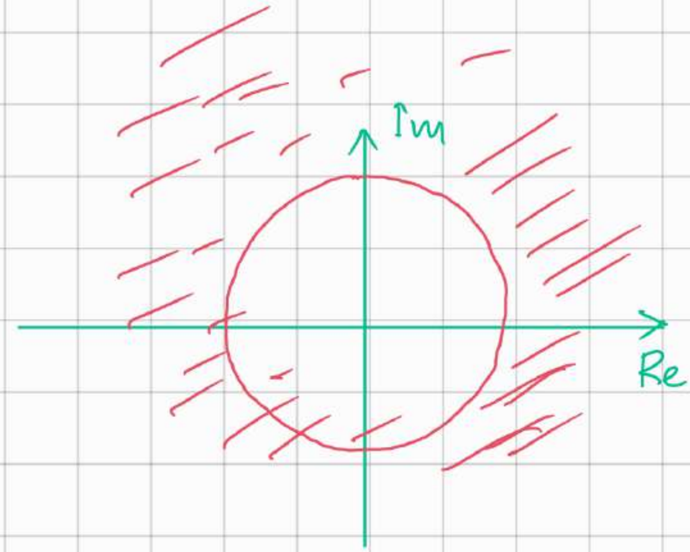


$z = x + iy$

$Re(z) = x$

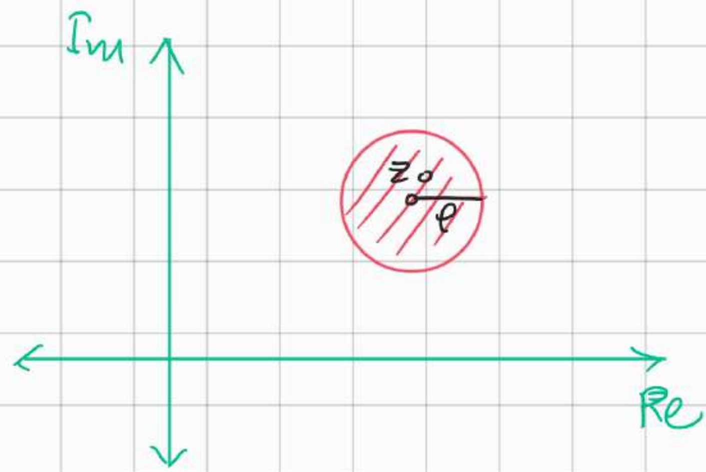
Example 2 :-

region : $|z| > 5$



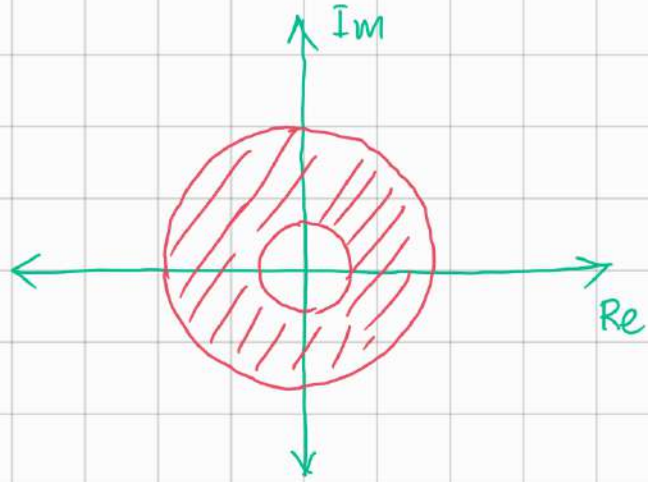
Example 3 :-

region : $|z - z_0| < \rho$



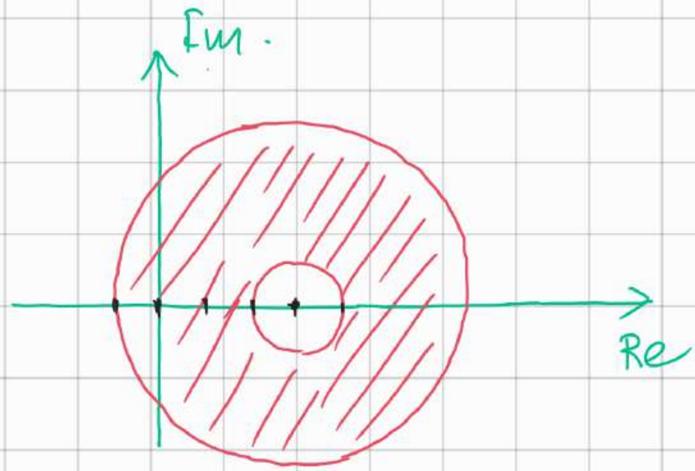
Example 4 :-

region : $1 < |z| < 5$



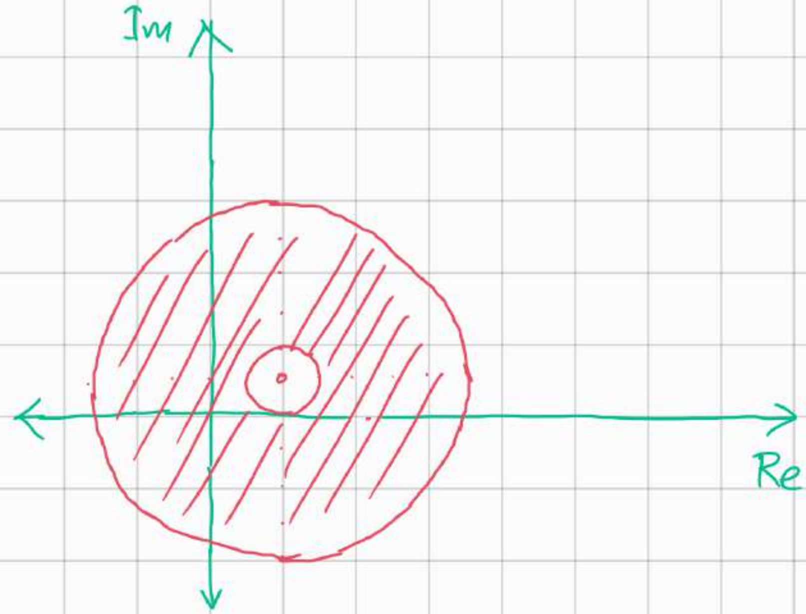
Example 5 :-

region : $1 < |z-3| < 5$



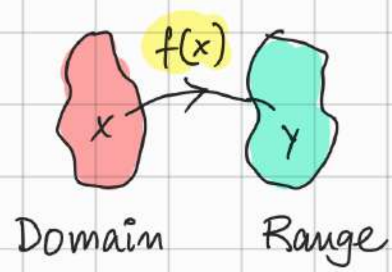
Example 6 :-

region : $1 < |z-(2+i)| < 5$



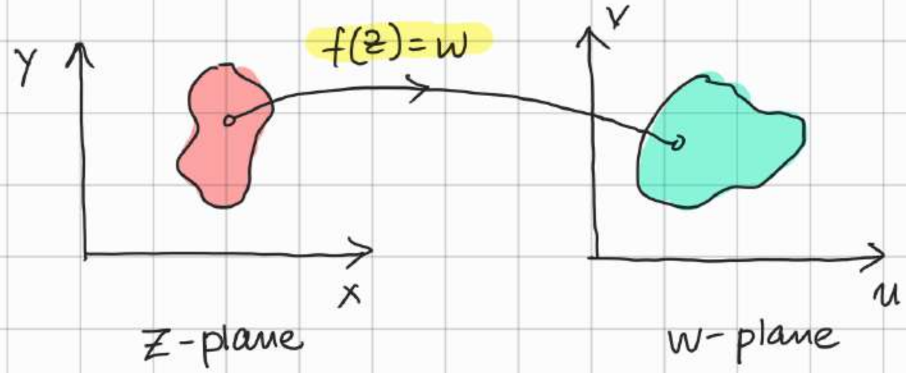
COMPLEX FUNCTION

Real function :



$y = f(x)$.

Complex function.



$$z = x + iy$$

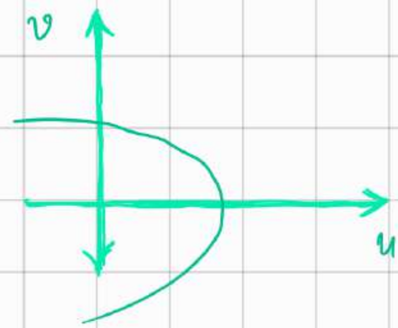
$$\Big\downarrow f(z)$$

$$w = u(x,y) + iv(x,y)$$

Example 1:- $f(x) = z^2$; $z = x + iy$

$$w = z^2 = \underbrace{(x^2 - y^2)}_{u} + i\underbrace{(2xy)}_{v}$$

$$u = x^2 - y^2 \quad ; \quad v = 2xy$$



Example 2:- $f(z) = z^2 - \operatorname{Im}(z + i)$

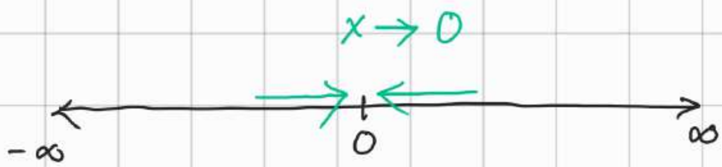
$$= (x^2 - y^2) + i\,2xy - (y + 1)$$

$$= (x^2 - y^2 - y - 1) + i\,2xy$$

CONCEPT OF LIMIT

What does $z \to 0$ mean?

Real case :-

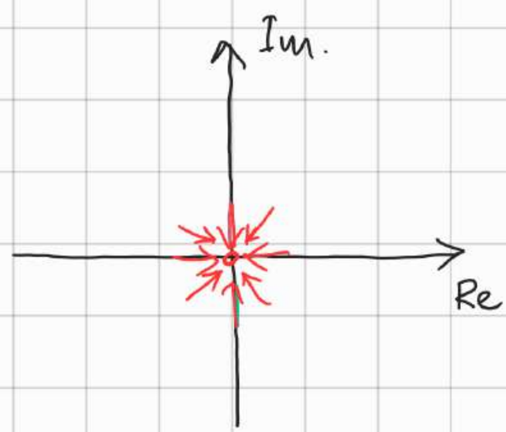


$\lim_{x \to x_0} f(x) = L$ means.

$$|f(x) - L| < \delta$$

$$\text{if } |x - x_0| < \epsilon_0$$

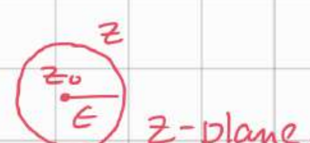
Complex case :-

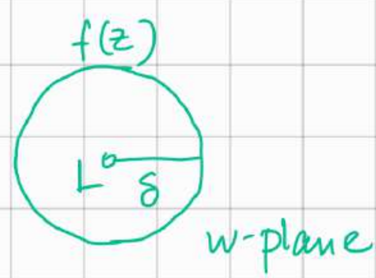


$\lim_{z \to z_0} f(z) = L$ means.

$$|f(z) - L| < \delta$$

$$\text{for } |z - z_0| < \epsilon$$





direction of approach : 'infinite'

When does limit exist ?

CONTINUITY

DERIVATIVE OF A COMPLEX FUNCTION

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

if $z = x + iy$

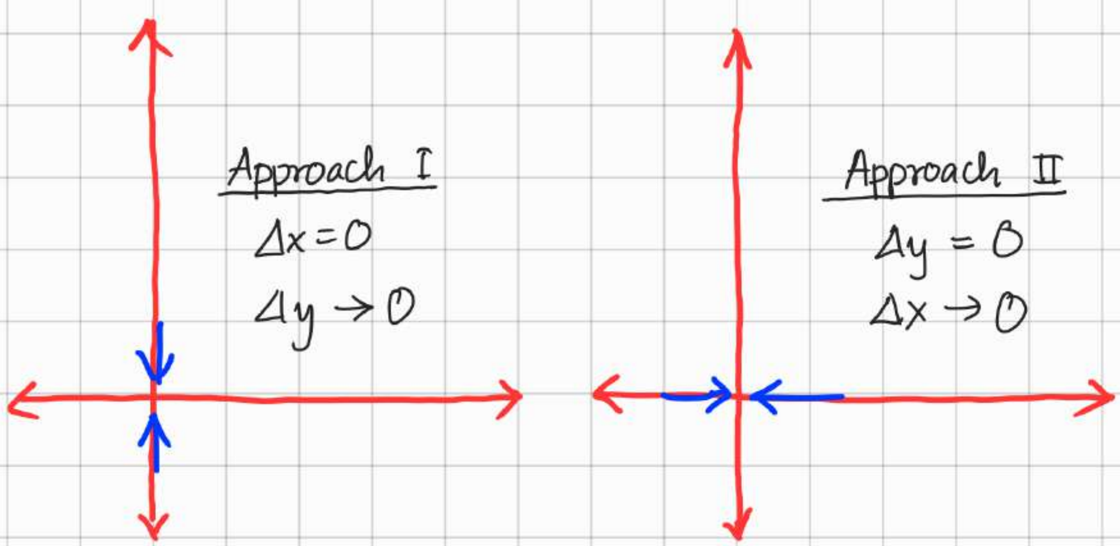
then $\Delta z = \Delta x + i\Delta y$

$\Delta z \to 0$ means $\Delta x + i\Delta y \to 0$

i.e. $\Delta x \to 0$

and $\Delta y \to 0$

Let us consider two approaches



[Although there are infinitely many approaches but C-R eqⁿ makes it app. independent]

Suppose, $f(z) = u(x,y) + iv(x,y)$

$$\therefore \Delta f(z) = \Delta u(x,y) + i\Delta v(x,y)$$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

Approach I: $\Delta x = 0 \,;\, \Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Approach II :- $\Delta y = 0 \,;\, \Delta x \to 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta y}{\Delta y}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

If derivative exists then.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;;\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann Condition.

ANALYTIC FUNCTIONS

$f(z)$ is an analytic function if :-

1. $f(z)$ obeys Cauchy-Riemann conditions.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \; ; \; \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Polar form:- $\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}} \; ; \; \boxed{r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}}$

2. $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial z}$ are continuous over a region.

Tip: If a function $f(z)$ is not explicitly a function of $\overline{z}$, then it is analytic.

i.e. $\frac{\partial f}{\partial \overline{z}} = 0$

Example 1:-

$$f(z) = z^2$$
$$= \underbrace{(x^2 - y^2)}_{u} + i\,\underbrace{2xy}_{v}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}.$$

Hence, the function is analytic

Example 2:-

$$f(z) = Re(z).$$
$$= \underbrace{x}_{u}. \qquad v \to 0.$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

Hence, the function is not analytic

Example 3:-

$$f(z) = e^z$$
$$= e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x [\cos y + i \sin y]$$
$$= \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$$

$$\frac{\partial u}{\partial x} = \cos y\, e^x \; ; \; \frac{\partial v}{\partial y} = \cos y\, e^x$$

$$\frac{\partial u}{\partial y} = -\sin y\, e^x \; ; \; \frac{\partial v}{\partial x} = \sin y\, e^x$$

Since, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

Hence, the function is analytic

Entire function :-

If $f(z)$ is analytic for all $z < \infty$ then it is called entire function.

Examples :- (i) any polynomial function of z like z^2.

(ii) $\exp(z)$

(iii) trigonometric function such as $\sin(z)$; $\cos(z)$

* $f(z) = \frac{1}{z}$ is not entire as it is undefined at $z = 0$.

Another form of Cauchy-Riemann Equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \qquad\qquad \Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial u}{\partial x \partial y}$$

Since, x and y are linearly independent.

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 v = 0$$

Similarly, $\nabla^2 u = 0$

Thus, a function is analytic if.

$\nabla^2 u = 0$ and $\nabla^2 v = 0$

Cauchy-Riemann Equations also suggest that u and v are harmonic functions.

POLAR FORM OF CAUCHY-RIEMANN EQ^n

$$z = re^{i\theta}$$
$$= r\cos\theta + ir\sin\theta$$

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
$$\Rightarrow \frac{dx}{d\theta} = -r\sin\theta \quad ; \quad \frac{dy}{d\theta} = r\cos\theta$$

$u(x,y)$

$$\Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow \frac{du}{d\theta} = \frac{\partial u}{\partial x}\frac{dx}{d\theta} + \frac{\partial u}{\partial y}\frac{dy}{d\theta}$$

$$= r\left[\cos\theta\frac{\partial u}{\partial y} - \sin\theta\frac{\partial u}{\partial x}\right]$$

$v(x,y)$

$$\Rightarrow dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\Rightarrow \frac{dv}{d\theta} = \frac{\partial v}{\partial x}\frac{dx}{d\theta} + \frac{\partial v}{\partial y}\frac{dy}{d\theta}$$

$$= r\left[\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right]$$

Similarly,

$$\frac{du}{dr} = \frac{\partial u}{\partial x}\frac{dx}{dr} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dr}$$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$= \cos\theta\frac{\partial v}{\partial y} - \cos\theta\frac{\partial v}{\partial x}$$

$$\frac{dv}{dr} = \frac{\partial v}{\partial x}\frac{dx}{dr} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dr}$$

$$= \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y}$$

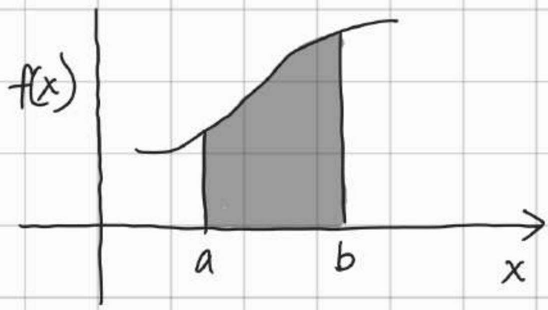
$$= \cos\theta\left(-\frac{\partial u}{\partial y}\right) + \sin\theta\frac{\partial u}{\partial x}$$

Thus, $\boxed{r\dfrac{\partial u}{\partial r} = \dfrac{dv}{d\theta}}$; $\boxed{r\dfrac{\partial v}{\partial r} = -\dfrac{\partial u}{\partial \theta}}$

Polar form of Cauchy-Riemann Equations

COMPLEX LINE INTEGRATION

→ Integration in real case ?



↳ area under the curve.

→ Integration in complex case ?

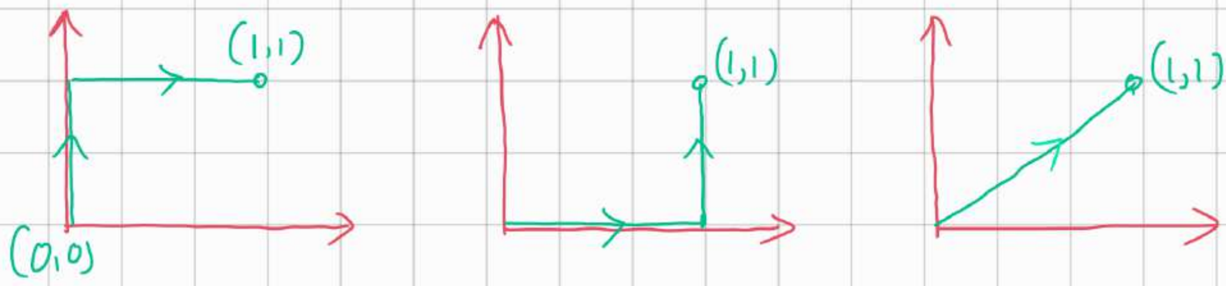
$z = x + iy$; x and y are two parameters which change independently when z changes.

→ we can perform the integration parametrically.

[For analytic function, integration is path-independent
For non-analytic function, integration depends on path.]

Example :- $f(z) = z^2$

three paths are given :-



Path I :- $\int_C f(z)\,dz = \int_C \{(x^2-y^2) + 2ixy\}\{dx + i\,dy\}$

$= -i\int_0^1 y^2\,dy + \int_0^1 \{(x^2-1) + 2ix\}\,dx$

$= -i\frac{y^3}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 - x\Big|_0^1 + 2i\frac{x^2}{2}\Big|_0^1$

$= -\frac{i}{3} + \frac{1}{3} - 1 + i$

$= -\frac{2}{3}(1-i)$

Path II :- $\int_C f(z)\,dz = \int_C \{(x^2-y^2) + 2ixy\}\{dx + i\,dy\}$

$= \int_0^1 x^2\,dx + \int_0^1 \{(1-y^2) + 2i.y\}\,i\,dy$

$= \frac{x^3}{3}\Big|_0^1 + iy\Big|_0^1 - i\frac{y^3}{3}\Big|_0^1 + 2i^2\frac{y^2}{2}\Big|_0^1$

$= \frac{1}{3} + i - \frac{i}{3} - 1 = -\frac{2}{3}(1-i)$

Path III :- $x = y$; $dx = dy$.

$$\int_C f(z)\,dz = \int_C \{(x^2-y^2) + 2ixy\}\{dx + i\,dy\}$$

$$= \int 2ix^2 (dx + i\,dx)$$

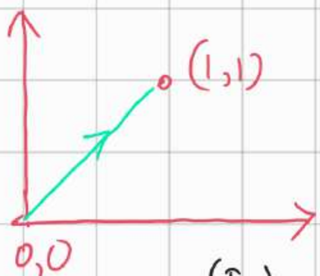
$$= 2i \int_0^1 x^2 (1+i)\,dx$$

$$= 2i \left.\frac{x^3}{3}\right|_0^1 + 2i^2 \left.\frac{x^3}{3}\right|_0^1$$

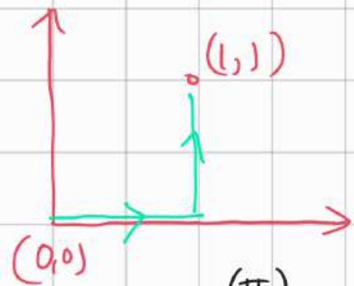
$$= -\frac{2}{3}(1-i)$$

Since, the function is analytic,
the result is path independent.

Check for function $f(z) = z^*$



(I)



(II)

$$z = x + iy$$
$$f(z) = z^* = x - iy.$$

Path I:

$$\int_{C_1} (x-iy)\{dx + idy\}$$

$$= \int_{C_1} (xdx + ydy) + i\int_{C_1} \cancelto{0}{(xdy - ydx)}$$

$x=y$

$$= 2\int_0^1 x\,dx$$

$$= 2\cdot\frac{x^2}{2}\Big|_0^1$$

$$= 1$$

Path II:

$$\int_0^1 x\,dx + \int_0^1 (1-iy)\,i\,dy$$

$$= \frac{x^2}{2}\Big|_0^1 + \int_0^1 (i+y)\,dy$$

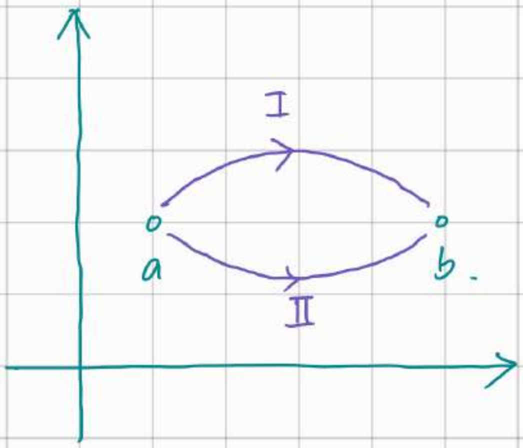
$$= \frac{1}{2} + i + \frac{1}{2}$$

$$= 1+i$$

Since, z^* is not analytic hence the result is path dependent.

CAUCHY-GOURSAT THEOREM.

Suppose $f(z)$ is analytic.



So, $\int_a^b f(z)\,dz = \int_a^b f(z)\,dz$

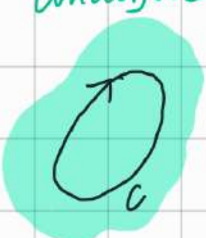
$\Rightarrow \int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$

$\Rightarrow \int_a^b f(z)\,dz + \int_b^a f(z)\,dz = 0$

$\Rightarrow \boxed{\oint f(z)\,dz = 0}$ Cauchy-Goursat Equation

---

analytic region.



Suppose we have a closed contour in an analytic region.

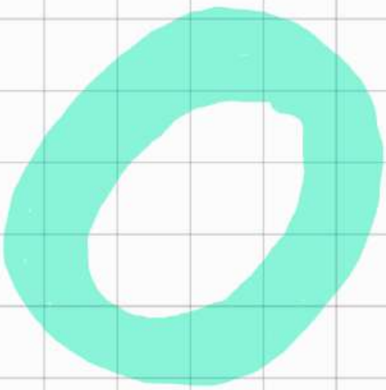
$$\oint f(z)\,dz = 0 \Rightarrow \oint (u+iv)(dx+idy)$$

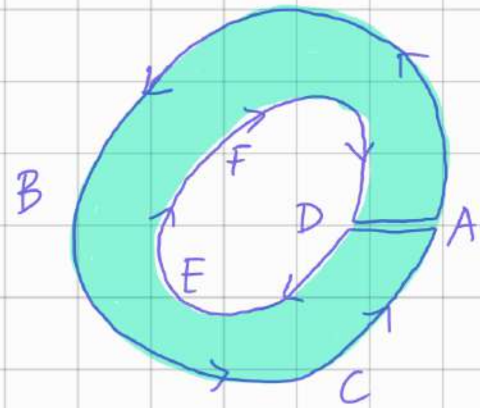
$$\Rightarrow \oint (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy)$$

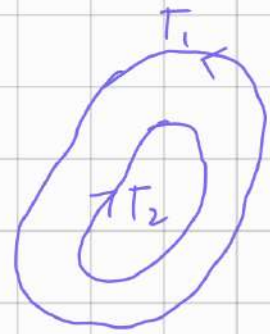
How to convert this closed loop integral to simple integral?

use :- $\boxed{\oint_c (P\,dx + Q\,dy) = \iint \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dy\,dx}$

Cauchy-Goursat Theorem holds for multiply connected regions as well.







We may convert any multiply connected region to a simply connected region by drawing a narrow channel.

since $$\int_{AD} f(z)\,dz = -\int_{DA} f(z)\,dz$$

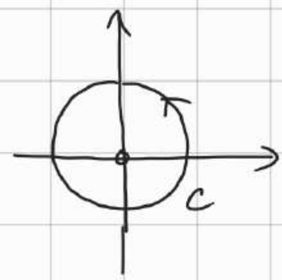
$$\therefore \oint_{\Gamma_1} f(z)\,dz = -\oint_{\Gamma_2} f(z)\,dz$$

$$\Rightarrow \oint_{\Gamma_1} f(z)\,dz = \oint_{\Gamma_2} f(z)\,dz$$

APPLICATIONS

$f(z) = \frac{1}{z}$

$\oint \frac{1}{z}\, dz = ?$ for $|z| < 1$



$z = re^{i\theta}$

$dz = (dr + ir\,d\theta)\, e^{i\theta}$

* At $z = 0$; there is singularity

$$\oint \frac{1}{z}\, dz = \oint \frac{1}{re^{i\theta}} (dr + ir\,d\theta) e^{i\theta}$$

$$= \oint \frac{dr}{r} + i \oint d\theta$$

$$= 0 + i\, 2\pi$$

$$= 2\pi i$$

CAUCHY'S INTEGRAL FORMULA

The Cauchy-Goursat theorem holds for calculation of integrals when the function is analytic within a region. What to do when the function is not analytic ?

Statement :-

$$\boxed{f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz}$$

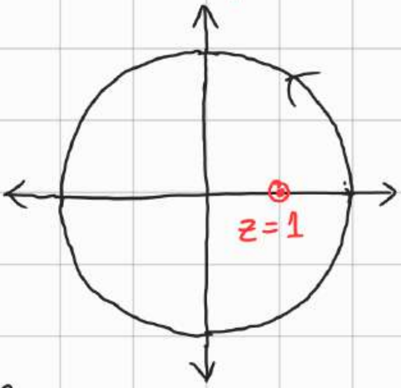
$$= \frac{1}{2\pi i} \oint g(z)\, dz$$

if $g(z)$ is not analytic but $f(z)$ is analytic in the region.

$$\boxed{f^{(n)}(a) = \frac{1}{2\pi i} (n!) \oint \frac{f(z)}{(z-a)^{n+1}}\, dz}$$

* a is a singularity in the region where $f(z)$ is analytic.

Example 1:- $\oint \frac{e^z(z^2+1)}{(z-1)^2}\,dz$ over $|z|=2$.



$f(z) = e^z(z^2+1)$

↳ analytic

$g(z) = \frac{f(z)}{(z-1)^2} \rightarrow$ not analytic

Here $n+1=2 \Rightarrow n=1$; $a=1$

$$\therefore \frac{2\pi i\, f'(a)}{1!} = \oint \frac{e^z(z^2+1)}{(z-1)^2}$$

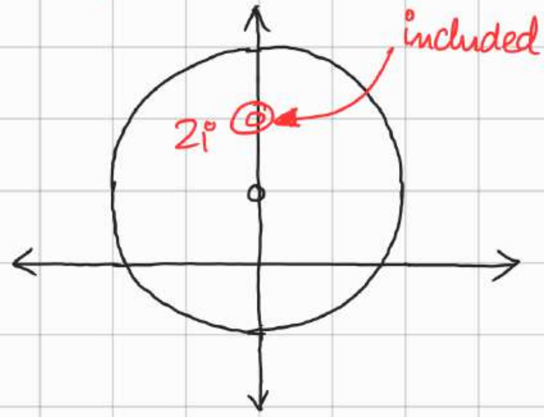
$$= 2\pi i \left[e^z\, 2z + (z^2+1)e^z \right]_{z=a}$$

$$= 2\pi i \left[e^1 (2a + a^2+1) \right]_{a=1}$$

$$= 2\pi i e \cdot 4$$

$$= 8\pi i e$$

Example 2:- $\oint \frac{z^2}{z^2+4}\,dz$ for $|z-i|=2$.



$$\oint \frac{z^2}{(z+2i)(z-2i)}$$

$$f(z) = \frac{z^2}{(z+2i)}$$

$a = 2i$ $\quad\quad$ $n = 0$.

$$\oint \frac{z^2}{z^2+4} = \frac{2\pi i}{0!} \cdot f(a).$$

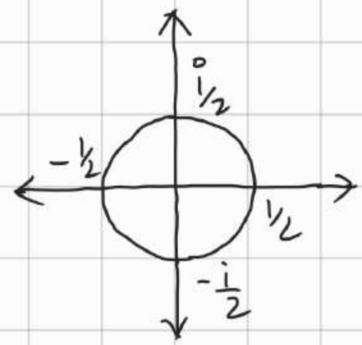
$$= 2\pi i \frac{(2i)^2}{(2i+2i)}$$

$$= \frac{2\pi i}{4i} \cdot -4$$

$$= -2\pi \quad //$$

Example 3:- $\oint \frac{2z+5}{(z^2-2z)}$ for $|z| = \frac{1}{2}$

$\oint \frac{2z+5}{(z^2-2z)}$



$= \oint \frac{2z+5}{z(z-2)}$

z is included.

$\therefore f(z) = \frac{2z+5}{z-2}$ | $z - a = z \Rightarrow a = 0.$ | $n+1 = 1 \Rightarrow n = 0.$

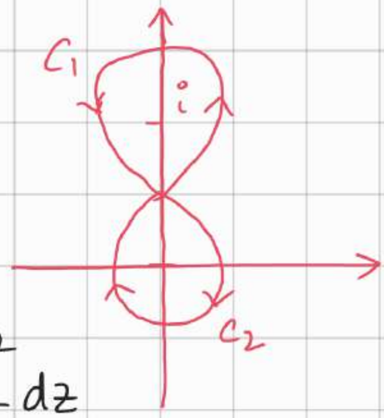
$\therefore \oint \frac{2z+5}{z(z-2)} = \frac{2\pi i}{0!} f(0)$

$= \frac{2\times 0+5}{0-2} \cdot 2\pi i$

$= -\frac{5}{2}\, 2\pi i$

$= -5\pi i$

Example 4:- $\oint \frac{z^3+3}{z(z-i)^2}$



$$\oint_{C_1} \frac{(z^3+3)/z}{(z-i)^2}\,dz + \oint_{C_2} \frac{(z^3+3)/(z-i)^2}{z}\,dz$$

$$= \oint_{C_1} \frac{(z^3+3)/z}{(z-i)^2}\,dz - \oint_{C_2} \frac{(z^3+3)/(z-i)^2}{z}\,dz$$

$$= \frac{2\pi i}{1!} f_1'(i) - \frac{2\pi i}{0!} f_2(0)$$

$$= 2\pi i \left(2z - \frac{3}{z^2}\right)_{z=i} + 2\pi i \cdot 3$$

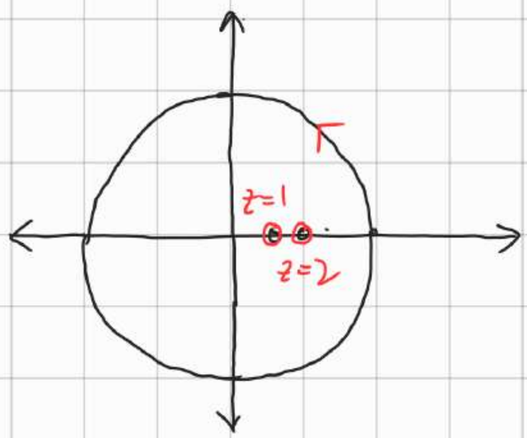
$$= 2\pi i (2i+3) + 6\pi i$$

$$= -4\pi + 6\pi i + 6\pi i$$

$$= -4\pi + 12\pi i$$

//

Example 5 :- $\oint_C \dfrac{z}{(z-1)(z-2)}\,dz \; ; \; |z| = 4$



$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$z = A(z-2) + B(z-1).$$

$$\Rightarrow z = Az + Bz - 2A - B.$$

$-2A - B = 0$ | $A + B = 1$

$\Rightarrow 2A = -B$ | $\Rightarrow B = 1 - A.$

$\therefore$

$2A = -(1-A)$

$\Rightarrow 2A = A - 1$

$\Rightarrow A = -1$

$\therefore B = 1 - (-1) = 2.$

$$\oint_C \frac{z}{(z-1)(z-2)}\,dz = \oint \frac{-1}{(z-1)}\,dz + \oint \frac{2}{(z-2)}\,dz.$$

$$= \frac{2\pi i}{0!}(-1) + \frac{2\pi i}{0!}\cdot 2.$$

$$= 4\pi i - 2\pi i$$

$$= 2\pi i$$

Also possible,

$$\oint_C = \frac{z/z-2}{z-1}\,dz + \oint_C \frac{z/z-1}{z-2}\,dz$$

$$= 2\pi i\, f_1(1) + 2\pi i\, f_2(2).$$

$$= 2\pi i \left[\frac{1}{1-2} + \frac{2}{2-1} \right]$$

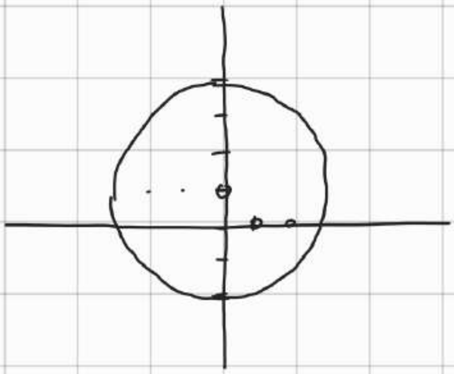
$$= 2\pi i \left[-1 + 2 \right]$$

$$= 2\pi i$$

Example :- $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ over $|z-i| = 3.$

$$\oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)/z-1}{z-2}\,dz$$

$$+ \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)/z-2}{z-1}$$



$$= 2\pi i \left(\frac{1}{1} + \frac{-1}{-1} \right)$$

$$= 4\pi i$$

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SEQUENCE & SERIES

A complex sequence is a mapping $f : \mathbb{N} \to \mathbb{C}$

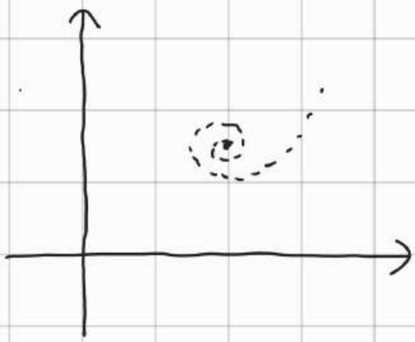
Domain : $\mathbb{N}$
Rance : $\mathbb{C}$

Eg :- $\{1 + i^n\} \to$ for $n = 1$: $1 + i$
for $n = 2$; 0
for $n = 3$; $1 - i$
for $n = 4$; 2
for $n = 5$; $1 + i$
- . . . .

Convergence of a sequence :

$\lim_{n\to\infty} z_n = L$; $|z_n - L| < \epsilon$.

in that case the sequence is converging.



If $L = a + ib$. ; $\lim_{n\to\infty} \mathrm{Re}(z_n) = a$

$\lim_{n\to\infty} \mathrm{Im}(z_n) = b$

Series: Summation of the sequence

$$\sum_n z_n = z_1 + z_2 + \dots + z_n$$

Convergence of a Series :-

If $\lim_{n\to\infty} \left(\sum z_n\right) = L$ then the series is said to be converging.

Converging geometric series :-

$$\sum_{n=1}^{\infty} a z^{n-1} = a + az + az^2 + \dots + az^{n-1} + az^n + \dots$$

$$S_n = a + az + az^2 + \dots + az^{n-1}$$

$$z S_n = az + az^2 + \dots + az^{n-1} + az^n$$

$$S_n - zS_n = a - az^n$$

$$\boxed{S_n = \frac{a - az^n}{1 - z}}$$

if the series is converging $z^n \to 0$

then $\boxed{S_n = \frac{a}{1 - z}}$

Example 1:- $\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k}$

$$a = \frac{(1+2i)}{5} \qquad z = \frac{1+2i}{5}$$

$$|z| = \sqrt{\frac{1^2+2^2}{5^2}} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}}$$

since $|z| < 1$

$$\therefore \sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{a}{1-z}$$

$$= \frac{1+2i}{5} \cdot \frac{1}{1-\frac{1+2i}{5}}$$

$$= (1+2i) \frac{1}{4-2i}$$

$$= \frac{(1+2i)(2+i)}{2(2-i)(2+i)}$$

$$= \frac{2+i+4i+2i^2}{2(4+1)}$$

$$= \frac{5i}{10}$$

$$= \frac{i}{2}$$

//

CONVERGENCE TEST

1. Ratio Test

Calculate $\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$

- If $L < 1$; the series is convergent
- If $L > 1$; the series is divergent
- If $L = 1$; no conclusion.

2. Root Test

Calculate $\lim_{n \to \infty} \sqrt[n]{|z_n|} = L$

- If $L < 1$; the series is convergent
- If $L > 1$; the series is divergent
- If $L = 1$; no conclusion.

CIRCLE AND RADIUS OF CONVERGENCE

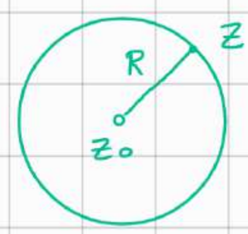
Power series :-

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

$$= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots + a_n (z - z_0)^n + \dots$$

It may be possible to express the power series in this region (circle)

$$|z - z_0| < R.$$



$R \rightarrow$ called the radius of convergence.

How to find out the circle of convergence ?

Ratio test :

Calculate: $\lim\limits_{k\to\infty} \left| \frac{z_{k+1}}{z_k} \right| = L$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{a_{k+1}(z - z_0)^{k+1}}{a_k (z - z_0)^k} \right| = L$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| |z - z_0| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| R < 1$$

Let $\lim\limits_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \ell.$

$\therefore \ell R < 1$

$\Rightarrow R < \frac{1}{\ell}$

max limit of R.

Example 1:- $\displaystyle\sum_{k=1}^{\infty} \frac{z^{k+1}}{k} = z^2 + \frac{z^3}{3} + \frac{z^4}{4} + \dots$

Ratio test. $\displaystyle l = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$

$$= \lim_{k\to\infty} \left|\frac{1/k+1}{1/k}\right|$$

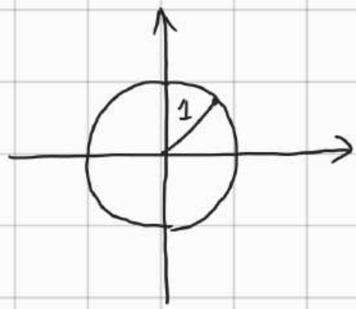
$$= \lim_{k\to\infty} \left|\frac{k}{1+k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{1}{\frac{1}{k}+1}\right|$$

$$= 1$$

$$Rl < 1 \Rightarrow R < \frac{1}{l}$$

i.e. $R < 1$



Example 2 :- $\displaystyle\sum_{k=0}^{\infty} \frac{(z-2i)^k}{(1-2i)^{k+1}}$

$$a_k = \frac{1}{(1-2i)^{k+1}} \qquad z_0 = 2i$$

$$l = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

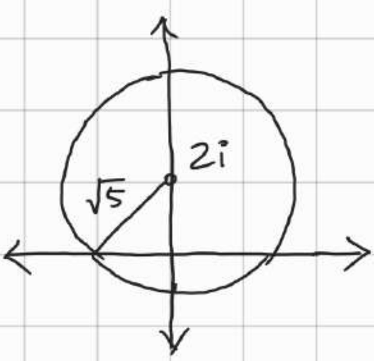
$$= \lim_{k\to\infty} \left| \frac{(1-2i)^{k+1}}{(1-2i)^{k+2}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{1}{1-2i} \right|$$

$$= \left| \frac{1+2i}{1+4} \right|$$

$$= \sqrt{\frac{1}{5^2} + \frac{2^2}{5^2}}$$

$$= \frac{1}{\sqrt{5}}$$



$\therefore$ $Rl < 1$

$\Rightarrow R < \frac{1}{l}$

i.e. $R \leq \sqrt{5}$

Example 3 :- $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k (z-1-i)^k}{k\, 2^k}$

$$z_0 = 1+i$$

$$a_k = \frac{(-1)^k}{k\, 2^k}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{(k+1)\, 2^{k+1}} \cdot \frac{k\, 2^k}{(-1)^k} \right|$$

$$= \lim_{k\to\infty} \frac{1}{2} \left| \frac{k}{k+1} \right|$$

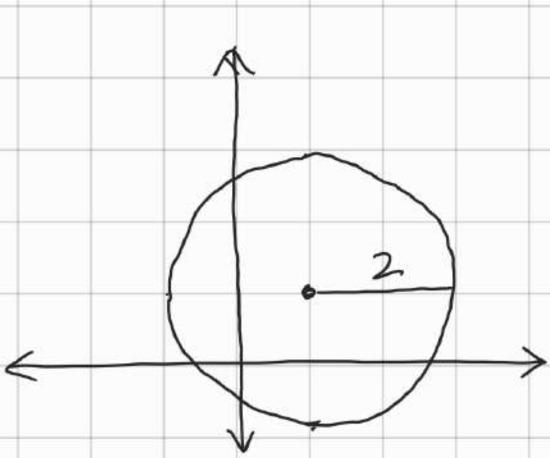
$$= \lim_{k\to\infty} \frac{1}{2} \left| \frac{1}{\frac{1}{k}+1} \right|$$

$$= \frac{1}{2}.$$

$$Rl < 1$$

$$\Rightarrow R \cdot \frac{1}{2} < 1$$

$$\Rightarrow R < 2.$$



TAYLOR SERIES OF COMPLEX FUNCTION

If $f(z)$ is analytic in some domain D

then $f(z)$ can be expanded as a power series around some point z_0

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

According to Cauchy's Integral formula,

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

If $z_0 = 0$, then

$$f(z) = \sum \frac{f^{(k)}(0)}{k!} \cdot z^k$$

$\rightarrow$ Maclaurin Series.

Example :- Expand $f(z) = \frac{1}{1-z}$ around $z = i$

$$f(z) = \frac{1}{(1-z)}$$

$$= \frac{1}{1-z+i-i} \quad [z_0 = i]$$

$$= \frac{1}{(1-i)-(z-i)}$$

$$= \frac{1}{1-i}\left[1 - \frac{z-i}{1-i}\right]^{-1}$$

$$= \frac{1}{1-i}\left[1 + \frac{z-i}{1-i} + \frac{(z-i)^2}{(1-i)^2} + \cdots\right]$$

$$= \sum_{k=0}^{\infty} \frac{(z-i)^k}{(1-i)^{k+1}}$$

$$= \sum a_k (z-i)^k$$

$$a_k = \frac{1}{(1-i)^{k+1}}$$

$$l = \lim_{k \to 0} \left|\frac{1}{1-i}\right| = \frac{1}{\sqrt{2}} \qquad \therefore \quad R = \sqrt{2}.$$

Example 2:- $f(z) = \frac{1}{3-z}$ around $z_0 = 2i$

$$f(z) = \frac{1}{3-z+2i-2i}$$

$$= \frac{1}{(3-2i)-(z-2i)}$$

$$= \left[\frac{(3-2i)-(z-2i)}{1}\right]^{-1}$$

$$= \frac{1}{3-2i}\left[1 - \frac{z-2i}{3-2i}\right]^{-1}$$

$$= \frac{1}{3-2i}\left[1 + \frac{z-2i}{3-2i} + \frac{(z-2i)^2}{(3-2i)^2} + \cdots\right]$$

$$= \sum_{k=0}^{\infty} \frac{(z-2i)^k}{(3-2i)^{k+1}}$$

Thus, $a_k = \frac{1}{(3-2i)^{k+1}}$

//

Example 3 :- $f(z) = \dfrac{z-1}{3-z}$ around $z_0 = 1$

$$f(z) = \frac{z-1}{3-z+1-1}$$

$$= \frac{z-1}{(3-1)-(z-1)}$$

$$= \frac{z-1}{2-(z-1)}$$

$$= \frac{z-1}{2}\left[1 - \frac{z-1}{2}\right]^{-1}$$

$$= \frac{z-1}{2}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{4} + \cdots\right]$$

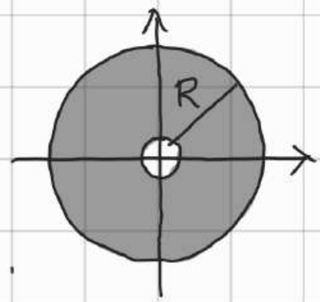
$$= \sum_{k=0}^{\infty} \frac{(z-1)^{k+1}}{2^{k+1}}$$

$$a_k = \frac{z-1}{2^{k+1}} \qquad z_0 = 1.$$

SINGULARITY

Suppose $f(z)$ is not analytic at z_0 but analytic in the neighbourhood of z_0 then z_0 is called a singular point.

Eg. $f(z) = \frac{1}{z}$



$0 < |z| < R$

$f(z)$ is analytic
but at $z=0$ it is blowing up.

There is a singularity at $z=0$.

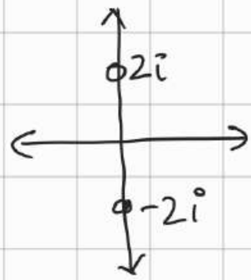
Eg:- $f(z) = \frac{1}{z-i}$ $\rightarrow$ singularity at i

SINGULARITIES

(i) Isolated Singularity
(ii) Removable Singularity
(iii) Essential Singularity

ISOLATED SINGULARITY

Eg:- $f(z) = \dfrac{1}{z^2+4} = \dfrac{1}{(z+2i)(z-2i)}$



$f(z)$ blows up at $z = \pm 2i$

$$\lim_{z \to 2i} (z-2i) f(z) = \frac{1}{2i+2i} = \frac{1}{4i} \neq 0.$$

also, $$\lim_{z \to -2i} (z+2i) f(z) = \frac{1}{z-2i} = \frac{1}{-4i} \neq 0$$

if $$\lim_{z \to z_0} (z-z_0)^n f(z) \neq 0$$

then $f(z)$ has singularity of order 'n'
or pole of the singularity is 'n'

Eg: $f(z) = \dfrac{1}{(z-4)^2(z-i)(z+3i)}$

$z = 4$ is a pole of order '2'.

* if $n = 1$; it is called a "simple pole"

REMOVABLE SINGULARITY

eg:- $\frac{\sin z}{z}$ $z = 0$ is a singularity

but $\lim_{z \to 0} \frac{\sin z}{z} = 1$

thus, the singularity is removed.

ESSENTIAL SINGULARITY

↳ singularity that cannot be removed.

eg:- $f(z) = e^{1/z}$

there is a singularity at $z = 0$ which we cannot remove.

LAURENT SERIES

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$= \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

$$= \underbrace{\cdots\cdots \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{(z-z_0)}}_{\text{principal part}}$$

$$+ \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\text{analytic part}}$$

If we have principal terms in the Laurent series expansion of a series, there will be singularities.

* If there is no principal term we may have removable singularities.

Example 1:- $f(z) = \dfrac{\sin z}{z}$.

$$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^7}{7!}$$

no principal terms.

↳ removable singularity

Example 2:- $f(z) = \dfrac{1}{z^3}\left[z - \sin z\right]$

$$f(z) = \frac{1}{z^3}\left[z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^7}{7!} - \cdots$$

no principal terms.

↳ removable singularity

Example 3:- $f(z) = \dfrac{1}{z^2}(\sin z)$.

$$f(z) = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \underline{\frac{1}{z}} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^7}{7!} + \cdots$$

↳ isolated singularity of order '1'.

Example 4 :- $f(z) = \dfrac{e^{2z}}{(z-1)^3}$

$$f(z) = \frac{e^{2(m+1)}}{m^3}$$

$$= \frac{e^2}{m^3} \cdot e^{2m}$$

$$= \frac{e^2}{m^3}\left[1 + 2m + \frac{(2m)^2}{2!} + \frac{(2m)^3}{3!} + \cdots\right]$$

$$= e^2\left[\frac{1}{m^3} + \frac{2}{m^2} + \frac{2}{m} + \frac{2^3}{3!} + \frac{2^4 m}{4!} + \cdots\right]$$

$$= e^2\left[\underbrace{\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)}}_{\text{principal terms.}} + \frac{2^3}{3!} + \frac{2^4(z-1)}{4!} + \cdots\right]$$

Example 5 :- $\dfrac{1}{(z-1)(z-2)}$ expand in $1<|z|<2$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$\underset{\Rightarrow \frac{|z|}{2}<1}{|z|<2} \qquad \underset{\Rightarrow \frac{1}{|z|}<1}{|z|>1}$$

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$= \underbrace{-\frac{1}{z^4}-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}}_{\text{principal part}} \; \underbrace{-\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\cdots}_{\text{analytic part}}$$

Thus, we have essential singularities because of the principal part.

Example 6 :- $f(z) = \frac{1}{(z-3)}$ $1 < |z-4| < 4$.

$$= \left[\frac{1}{z-3} - \frac{1}{z}\right] \cdot \frac{1}{3}$$

$$= \frac{1}{3}\left[\frac{1}{(z-4)+1} - \frac{1}{(z-4)+4}\right]$$

$$= \frac{1}{3}\left[\frac{1}{(z-4)\left(1+\frac{1}{z-4}\right)}\right] - \frac{1}{3}\,\frac{1}{4\left[1+\frac{z-4}{4}\right]}$$

$$= \frac{1}{3(z-4)}\left(1+\frac{1}{z-4}\right)^{-1} - \frac{1}{12}\left[1+\left(\frac{z-4}{4}\right)\right]^{-1}$$

$$= \frac{1}{3}\,\frac{1}{z-4}\left[1 - \frac{1}{z-4} + \frac{1}{(z-4)^2} - \frac{1}{(z-4)^3} + \cdots\right]$$

$$- \frac{1}{12}\left[1 - \left(\frac{z-4}{4}\right) + \frac{(z-4)^2}{4^2} - \cdots\right]$$

Example:- $f(z) = \dfrac{2z-1}{z^3 - z^2}$ $0 < |z| < 1$.

$$f(z) = \frac{z + (z-1)}{z^2(z-1)}$$

$$= \frac{1}{z(z-1)} + \frac{1}{z^2}$$

$$= \frac{1}{z-1} - \frac{1}{z} + \frac{1}{z^2}$$

$$= \frac{1}{z^2} - \frac{1}{z} - (1-z)^{-1}$$

$$= \frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 - z^3 \ldots$$

in the interval $0 < |z-1| < 1$

$$f(z) = \frac{z + z - 1}{z^2(z-1)}$$

$$= \frac{1}{z(z-1)} + \frac{1}{z^2}$$

$$= \frac{1}{z-1} - \frac{1}{z} + \frac{1}{z^2}$$

$$= \frac{1}{z-1} - \frac{1}{(z-1)+1} + \frac{1}{z^2}$$

$$= \frac{1}{z-1} - \left[1 + (z-1)\right]^{-1} + \frac{1}{z^2}$$

$$= \frac{1}{z-1} - \sum^{\infty}(z-1)^k(-1)^k + \frac{1}{z^2}$$

$$\frac{1}{z^2} = -\frac{d}{dz}\left(\frac{1}{z}\right)$$

$$= -\frac{d}{dz}\left[\sum_{k=0}^{\infty}(z-1)^k(-1)^k\right]$$

$$= \sum_{k=0}^{\infty}(-1)^{k+1}k(z-1)^{k-1}$$

$$f(z) = \frac{1}{z-1} - \sum_{k=0}^{\infty}(z-1)^k(-1)^k + \sum(-1)^{k+1}k(z-1)^{k-1}$$

RESIDUE

Laurent Series Expansion of $f(z)$.

$$f(z) = \cdots + \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0} +$$

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots$$

co-efficient of the term $\frac{1}{z-z_0}$ i.e. a_{-1} is the 'residue'

- If the funct^n has a simple pole at $z = z_0$ then multiply the $f(z)$ by $(z-z_0)$

  such that $(z-z_0)f(z) = a_{-1}$ (others vanish)

$$\text{Residue}\left(f(z), z_0\right) = \lim_{z \to z_0} (z-z_0) f(z).$$

- If the function has a pole of order 'k' then.

$$\text{Residue}\left(f(z), z_0\right) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z).$$

Calculation of residue for quotient form

Suppose. $f(z) = \dfrac{g(z)}{h(z)}$

↳ $g(z)$ and $h(z)$ are both analytic

↳ say z_0 is a singular point

and $g(z_0) \neq 0$ but $h(z_0) = 0$
then we get a pole of order 1.

then $\boxed{\mathrm{Res}\left[f(z), z_0\right] = \dfrac{g(z_0)}{h'(z_0)} = \dfrac{g(z_0)}{\frac{d}{dz}(h(z))\Big|_{z=z_0}}}$

How?
$$h'(z) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0}$$

$$\Rightarrow h'(z) = \lim_{z \to z_0} \frac{h(z)}{z - z_0}$$

$$\Rightarrow \lim_{z \to z_0} h(z) = (z - z_0).$$

Thus, $$\mathrm{Res}\left[f(z), z_0\right] = \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to z_0} \frac{g(z)}{h(z)/z - z_0}$$

$$= \frac{g(z)}{h'(z_0)}\Bigg|_{z = z_0}$$

Example : Calculate residue for $f(z) = \frac{1}{(z-1)^2(z-3)}$

Singularities : $z=1$; $z=3$.

$z=1 \downarrow$ order 2 ; $z=3 \downarrow$ simple pole.

Residue at 3 :-

$$\text{Res}\left(f(z), 3\right) = \lim_{z \to 3} (z-3) \frac{1}{(z-1)^2(z-3)}$$

$$= \left.\frac{1}{(3-1)^2}\right|_{z=3}$$

$$= \frac{1}{4}$$

Residue at 1 :-

$$\text{Res}\left(f(z), 1\right) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}} (z-z_0)^2 \cdot \frac{1}{(z-z_0)^2(z-3)}$$

$$= \lim_{z \to 1} \frac{d}{dz}\left(\frac{1}{z-3}\right)$$

$$= -\frac{1}{(z-3)^2}$$

$$= -\frac{1}{4}$$

Example : calculate residue $f(z) = \dfrac{5z^2 - 4z + 3}{(z+1)(z+2)(z+3)}$

Singularities at, $-1, -2, -3$.

$$\text{Res}\left[f(z), -1\right] = \lim_{z \to -1} (z+1) \frac{5z^2 - 4z + 3}{(z+1)(z+2)(z+3)}$$

$$= \lim_{z \to -1} \frac{5z^2 - 4z + 3}{(z+2)(z+3)} = \frac{5+4+3}{1 \times 2} = 6$$

$$\text{Res}\left[f(z), -2\right] = \lim_{z \to -2} (z+2) \frac{5z^2 - 4z + 3}{(z+1)(z+2)(z+3)}$$

$$= \lim_{z \to -2} \frac{5z^2 - 4z + 3}{(z+1)(z+3)} = \frac{20+8+3}{(-1)(1)} = -31$$

$$\text{Res}\left[f(z), -3\right] = \lim_{z \to -3} \frac{5z^2 - 4z + 3}{(z+1)(z+2)} = \frac{45+12+3}{(-2)(-1)} = 30.$$

Example :- $f(z) = \dfrac{e^z}{z^3}$

$$f(z) = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \dots$$

Residue, $a_{-k} = \dfrac{1}{2}$

Example :- $f(z) = \dfrac{1}{\sin(\pi z)}$

$z_0 = 0 \quad ; \quad g(z_0) = 1$.

$h(z_0) = 0$.

$$\text{Res}\left[f(z), 0\right] = \left.\frac{1}{\pi \cos \pi z}\right|_{z=0} = \frac{1}{\pi}$$

singularities present for all 'n'.

$$\text{Res.}\left[f(z), n\right] = \left.\frac{1}{\pi \cos \pi z}\right|_{n}$$

$$= \frac{1}{\pi\ (-1)^n}$$

$$= \frac{(-1)^n}{\pi}$$

Example :- $f(z) = \dfrac{e^z}{e^z - 1}$

$$g(z) = e^z \;;\; h(z) = e^z - 1.$$

singularity at $z = 0$

$$g(0) \neq 0 \;;\; h(0) = 0$$

$$\text{Res.}\left[f(z), 0\right] = \frac{g(0)}{h'(0)}$$

$$= \frac{e^0}{e^0} = 1$$

Example :- $f(z) = \sec(z) = \dfrac{1}{\cos z}$

singularities at all $z = (2n+1)\pi/2$

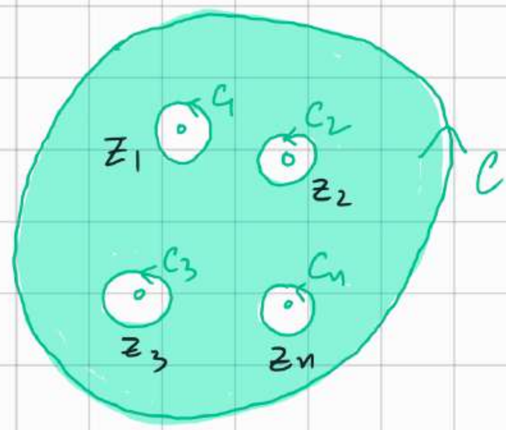
$$\text{Res}\left[f(z), (2n+1)\pi/2\right] = \left.\frac{1}{-\sin z}\right|_{(2n+1)\pi/2}.$$

$$= \frac{1}{-(-1)^n}$$

$$= -(-1)^n$$

$$= (-1)^{n+1}$$

CAUCHY'S RESIDUE THEOREM



Suppose $f(z)$ is an analytic function in the region bounded by curve C except at finite points $z_1, z_2, \ldots, z_n$

then

$$\boxed{\oint_C f(z)\,dz = \sum_{i=1}^{n} \oint_{C_i} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \text{Res}\left(f, z_i\right)}$$

If there is no singular point $\oint_C f(z)\,dz = 0$

How ??

$$f(z) = \sum_{n=-\infty}^{+\infty} a_k (z-z_0)^k$$

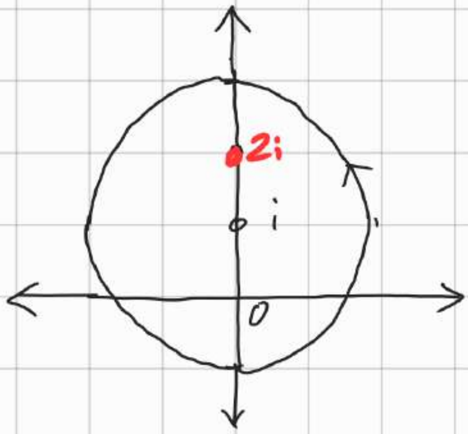
$$= \cdots + \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

For $k=-1$; $a_{-1} = \dfrac{1}{2\pi i} \oint_C f(z)\,dz$

$$\therefore \oint_C f(z)\,dz = 2\pi i\, a_{-1} = 2\pi i \,\text{Res}\left[f(z), z_0\right]$$

Example:- $\oint \frac{z+6}{(z^2+4)}\,dz \quad ; \quad |z-i|=2.$

$$\oint \frac{z+6}{(z+2i)(z-2i)}$$



$$\mathrm{Res}\left[f(z), 2i\right]$$

$$= \lim_{z \to 2i} \left(\frac{z+6}{z+2i} \right)$$

$$= \frac{2i+6}{4i}$$

$$= \frac{1}{2} - \frac{3}{2}i$$

$$\therefore \oint \frac{z+6}{z^2+4}\,dz = 2\pi i \,\mathrm{Res}\left[f(z), 2i\right]$$

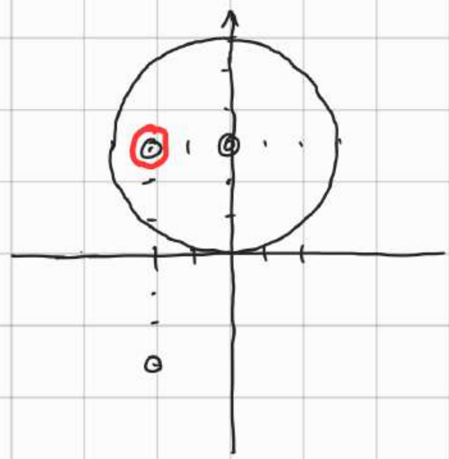
$$= 2\pi i \left[\frac{1}{2} - \frac{3}{2}i\right]$$

$$= \pi i + 3\pi$$

$$= \pi(i+3)$$

Example: $\oint \frac{1}{z^2+4z+13}\, dz$. $C: |z-3i| = 3$

$z^2 + 4z + 13 = 0$



$\Rightarrow z = \frac{-4 \pm \sqrt{4^2 - 4.13}}{2}$

$= -2 \pm \frac{\sqrt{16-52}}{2}$

$= -2 \pm 3i$

Singularity at $-2+3i$

$\mathrm{Res}\left[f(z), -2+3i\right] = \lim_{z \to z_0} (z-z_0) \frac{1}{(z-z_0)(z-z_1)}$

$= \lim_{z \to z_0} \frac{1}{z - z_1}$

$= \lim_{z \to z_0} \frac{1}{z - (-2-3i)}$

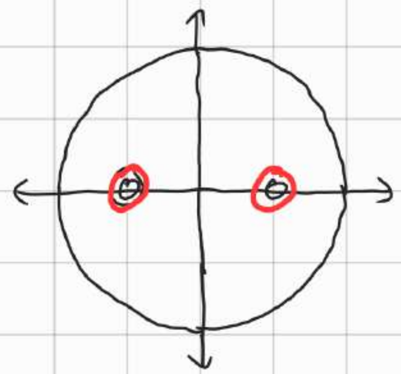
$= \frac{1}{-2+3i-(-2-3i)}$

$= \frac{1}{6i}$

$\therefore \oint \frac{1}{z^2+4z+13}\, dz = 2\pi i \frac{1}{6i} = \frac{\pi}{3}$

Example :- $\oint \frac{z e^z}{z^2-1}$ over $|z| = 2$.

$$\oint \frac{z e^z}{(z+1)(z-1)}$$



$\text{Res}\left[f(z), -1\right]$

$$= \lim_{z \to -1} \frac{z e^z}{z-1}$$

$$= \frac{-1 e^{-1}}{-2} = \frac{1}{2e}$$

$\text{Res}\left[f(z), +1\right]$

$$= \lim_{z \to +1} \frac{z e^z}{z+1}$$

$$= \frac{e}{2}$$

$$\therefore \oint \frac{z e^z}{z^2-1} = 2\pi i \left[\frac{1}{2e} + \frac{e}{2}\right]$$

$$= 2\pi i \left[\frac{e^{-1} + e^{1}}{2}\right]$$

$$= 2\pi i \cosh(1)$$

//

Example:- $\oint \frac{1}{(z-1)(z+2)^2}\, dz \qquad |z|=3$

sing. at $z=1$ and $z=-2$

$$\operatorname{Res}[f(z).1] = \lim_{z\to 1} \frac{1}{(z+2)^2} = \frac{1}{3^2} = \frac{1}{9}$$

$$\operatorname{Res}[f(z).-2] = \frac{1}{(2-1)!} \lim_{z\to -2} \frac{d^{2-1}}{dz^{2-1}}\left[(z+2)^2 \cdot \frac{1}{(z-1)(z+2)^2}\right]$$

$$= \lim_{z\to -2} \frac{d}{dz}\left(\frac{1}{z-1}\right)$$

$$= -\frac{1}{(z-1)^2}\bigg|_{-2} = -\frac{1}{9}$$

$$\therefore \oint \frac{1}{(z-1)(z+2)^2}\, dz = 2\pi i\left(\frac{1}{9} - \frac{1}{9}\right) = 0$$

Example :- $\oint \frac{z^3}{e^{1/z^2}}\, dz \qquad |z| = 1$

there is a singularity at $z = 0$.

$z^3 e^{-\frac{1}{z^2}}$

$= z^3\left(1 + \left(-\frac{1}{z^2}\right) + \frac{1}{2!}\cdot\left(-\frac{1}{z^2}\right)^2 + \frac{1}{3!}\left(-\frac{1}{z^2}\right)^3 + \cdots\right)$

$= \cdots \frac{1}{2}\cdot z^3 \cdot \frac{1}{z^2} \cdots$

$= \frac{1}{2z}$

Thus, residue is $\frac{1}{2}$.

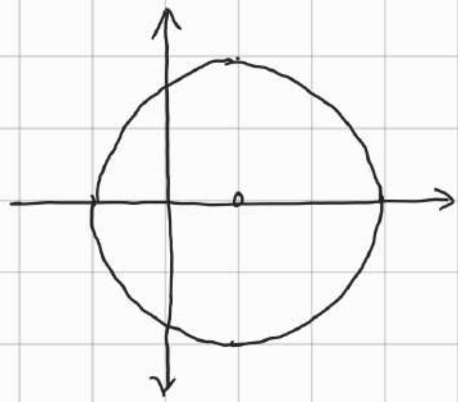
$\therefore \oint \frac{z^3}{e^{1/z^2}}\, dz = 2\pi i \cdot \frac{1}{2}$

$= \pi i$

Example :- $\oint \frac{\tan z}{z}\, dz \qquad |z-1| = 2.$

singularity at $z = 0$

$z = \frac{\pi}{2}$



$\frac{\tan z}{z}$

$= \frac{\sin z}{z \cos z} \qquad \left(\begin{array}{l} h' = \cos z - z \sin z \\ h'(0) = 1 \end{array} \right)$

at $z = 0$, $h(z) = 0$; $g(z) \neq 0$

$\therefore \operatorname{Res}[f(z), 0] = \frac{g(0)}{h'(0)} = \frac{0}{1} = 0$

at $z = \frac{\pi}{2}$; $h(z) = 0$; $g(z) \neq 0$

$h'(\pi/2) = \cos \pi/2 - \pi/2 \sin \pi/2 = -\pi/2$

$\therefore \operatorname{Res}[f(z), 0] = \frac{g(0)}{h'(0)} = -\frac{2}{\pi}$

$\therefore \oint \frac{\tan z}{z}\, dz = 2\pi i \left(-\frac{2}{\pi} \right) = -4i$

REAL INTEGRATION - CAUCHY RESIDUE THEOREM

Cauchy residue Theorem may be applied for.

(i) Trignometric functions.

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

(ii) Pure Algebraic functions.

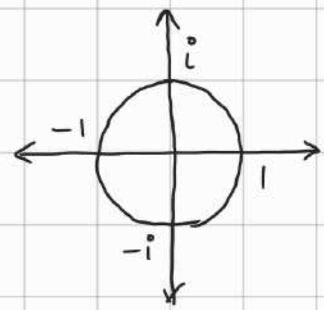
$$\int_{-\infty}^{+\infty} F(x)\, dx$$

(iii) Trigonometric + Algebraic functions

$$\int F(\cos x, \sin x, x)\, dx$$

TRIGONOMETRIC FUNCTION INTEGRATION.

- take unit circle
- take polar form.



$z = e^{i\theta}$ $[r = 1]$

$dz = iz\, d\theta$

$\Rightarrow$ $d\theta = \dfrac{dz}{iz}$

- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ i.e. $\cos\theta = \dfrac{z + \bar{z}}{2}$ or $\dfrac{z + \frac{1}{z}}{2}$

- $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ i.e. $\sin\theta = \dfrac{z - \bar{z}}{2i}$ or $\dfrac{z - \frac{1}{z}}{2i}$

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint F\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \frac{dz}{iz}$$

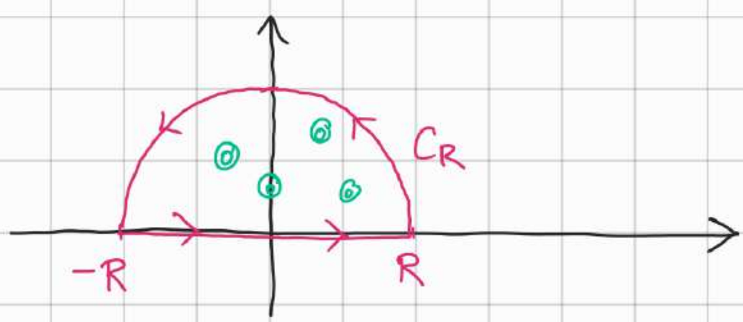
over unit circle.

ALGEBRAIC FUNCTION INTEGRATION

- $$\int_{-\infty}^{+\infty} F(x)\,dx = \lim_{R \to \infty} \int_{-R}^{+R} F(x)\,dx$$

  If this limit exists then the integral is convergent

- Consider the half-circle with radius R



- Change $F(x)$ to $F(z)$

- $$\oint f(z)\,dz = \int_{C_R} F(z)\,dz + \int_{-R}^{R} F(x)\,dx$$

  $$\Rightarrow 2\pi i \sum_{i=1}^{n} \text{Res}(F, z_i) = \int_{C_R} F(z)\,dz + \int_{-R}^{R} F(x)\,dx$$

Example :- $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

$$\oint \frac{dz/iz}{5+4\frac{\left(z-\frac{1}{z}\right)}{2i}}$$

$$\oint \frac{2i\,dz}{iz\left(10i+4z-\frac{4}{z}\right)}$$

$$= \oint \frac{dz}{z\left(5i+2z-\frac{2}{z}\right)}$$

$$= \oint \frac{dz}{\left(2z^2+5iz-2\right)}$$

$$2z^2+5iz-2=0$$

$$\Rightarrow z = \frac{-5i \pm \sqrt{(5i)^2-4\cdot 2\cdot(-2)}}{2\cdot 2}$$

$$-\frac{5}{4}i \pm \frac{\sqrt{-25+16}}{4}$$

$$= -\frac{5}{4}i \pm \frac{3i}{4}$$

$$= \frac{-2i}{4} \; ; \; \frac{-8i}{4}$$

$$= -\frac{i}{2} \; ; \; (-2i)_{\text{not included}}$$

singularity only at $\frac{-i}{2}$.

$$\oint \frac{dz}{2z^2+5iz-2}$$

$$\text{Res}\left[f(z), -\frac{i}{2}\right] = \lim_{z\to\frac{-i}{2}} \frac{1}{4z+5i} = \frac{1}{4\left(\frac{-i}{2}\right)+5i} = \frac{1}{3i}$$

$$\therefore \oint \frac{dz}{5+4\sin\theta} = 2\pi i \, \frac{1}{3i} = \frac{2\pi}{3}$$

//

Example:- $\int_0^{2\pi} \frac{1}{1+\frac{1}{2}\sin\theta}\, d\theta$

$$\oint \frac{dz}{iz\left(1+\frac{1}{2}\cdot\frac{z-\frac{1}{z}}{2i}\right)}$$

$$\oint \frac{dz}{iz\left(\frac{4i+z-\frac{1}{z}}{4i}\right)}$$

$$= \oint \frac{4i\ dz}{i\,(z^2+4iz-1)}$$

$$= \oint \frac{4\,dz}{z^2+4iz-1}$$

$$z^2+4iz-1=0$$

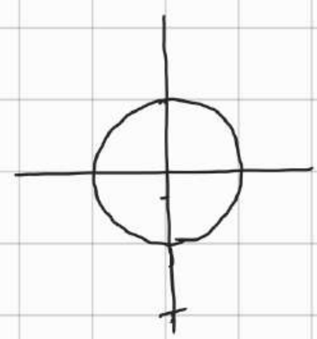
$$\Rightarrow z = \frac{-4i \pm \sqrt{4i^2-4(1)(-1)}}{2\cdot 1}$$

$$= -2i \pm \frac{\sqrt{-16+4}}{2}$$

$$= -2i \pm \frac{2\sqrt{3}\,i}{2}$$

$$= -2i \pm \sqrt{3}\,i$$

$$= -2i \pm 1.7\, i$$



included point $-2i+\sqrt{3}i$

$$\text{Res}\left[f(z), (\sqrt{3}-2)i\right] = \left.\frac{4}{2z+4i}\right|_{(\sqrt{3}-2)i}$$

$$= \frac{4}{2\sqrt{3}i-4i+4i}$$

$$= \frac{2}{\sqrt{3}i}$$

$$\therefore \oint \frac{4\,dz}{z^2+4iz-1} = 2\pi i\cdot\frac{2}{\sqrt{3}i} = \frac{4\pi}{\sqrt{3}} \quad //$$

Example :- $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\, d\theta$

$$\oint \frac{\frac{z+\frac{1}{z}}{2}}{3+\frac{z-\frac{1}{z}}{2i}} \frac{dz}{iz}$$

$$= \oint \frac{z+\frac{1}{z}}{6i+z-\frac{1}{z}} \cdot \frac{dz}{iz}$$

$$= \oint \frac{z^2+1}{z(z^2+6iz-1)}\, dz$$

$$z^2+6iz-1=0$$

$$\Rightarrow z = \frac{-6i}{2} \pm \frac{\sqrt{(6i)^2-4.(-1).1}}{2}$$

$$= -3i \pm \frac{1}{2}\sqrt{-36+4}$$

$$= -3i \pm \frac{1}{2}\sqrt{4^2.2}$$

$$= -3i \pm \frac{1}{2}.4.\sqrt{2}$$

$$= -3i \pm 2\sqrt{2}$$

singularity at $0, (2\sqrt{2}-3)i$

$$\text{Res}\left[f(z),0\right] = \lim_{z\to 0} \frac{z^2+1}{z^2+6iz-1} = \frac{1}{-1} = -1$$

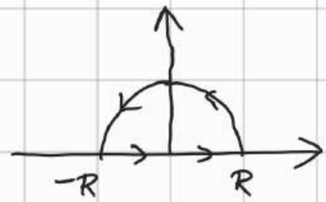
$$\text{Res}\left[f(z), (2\sqrt{2}-3)i\right] = \left.\frac{z^2+1/z}{2z+6i}\right|_{(2\sqrt{2}-3)i} = \frac{-(2\sqrt{2}-3)^2+1}{-(2\sqrt{2}-3)^2-6(2\sqrt{2}-3)}$$

$$= 1.$$

$$\therefore \oint_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\, d\theta = 2\pi i\,(-1+1) = 0$$

Example:- $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2-2x+2}\, dx.$

$$\hookrightarrow \oint \frac{1}{z^2-2z+2}\, dz$$



$$z^2-2z+2=0$$

$$\Rightarrow z = \frac{-(-2)}{2} \pm \frac{1}{2}\sqrt{(-2)^2-4.2.1}$$

$$= 1 \pm \frac{1}{2}\sqrt{-4}$$

$$= 1 \pm \frac{1}{2}.2i$$

$$= 1 \pm i$$

only $1+i$ will be included in the semi-circular region.

$$\therefore \oint \frac{1}{z^2-2z+2}\, dz = \int_{C_R} \frac{1}{z^2-2z+2}\, dz + \int_{-R}^{R} \frac{1}{x^2-2x+2}\, dx$$

$$\Rightarrow \int_{-R}^{R} \frac{1}{x^2-2x+2}\, dx = \oint \frac{1}{z^2-2z+2}\, dz - \int_{C_R} \frac{1}{z^2-2z+2}\, dz.$$

$$= \text{Res}\left[f(z), 1+i\right].2\pi i$$

$$\downarrow$$

Let $z = Re^{i\theta}$

$\Rightarrow dz = Rie^{i\theta}\, d\theta$

$\therefore$ the integral is $\displaystyle\int \frac{Rie^{i\theta}\, d\theta}{|R^2e^{i2\theta}-2Re^{i\theta}+2|}$

at $R \to \infty$ it goes to 0.

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2-2x+2}\, dx = \text{Res}\left[f(z), 1+i\right]$$

$$= 2\pi i \left. \frac{1}{2z-2} \right|_{1+i}$$

$$= 2\pi i \frac{1}{2+2i-2}$$

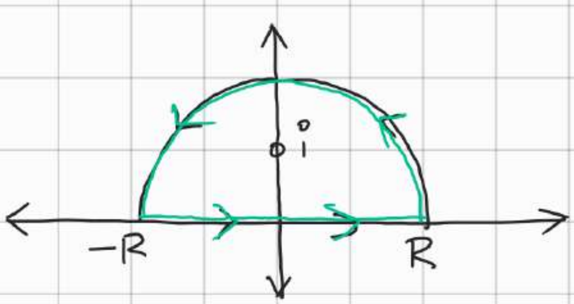
$$= \frac{2\pi i}{2i}$$

$$= \pi \quad //.$$

Example :- $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x\, dx}{x^2+1}$

When there is a mixture of trig. and algebraic, convert $x \to z$.

$$\oint \frac{e^{iz}}{z^2+1}\, dz \qquad \left[\text{convert trig. to exponential}\right]$$



$$e^{iz} = e^{i(x+iy)} = e^{-y}\left[\cos x + i\sin x\right] = \cos x + i\sin x$$

since over the real axis $y=0$

Now, $z^2+1=0 \Rightarrow z = \pm i$

We shall include only $z=+i$ in the semi-circular region.

$$\oint \frac{e^{iz}}{z^2+1}\,dz = \int_C \frac{e^{iz}}{z^2+1}\,dz + \int_{-R}^{R} \frac{\cos x}{x^2+1}\,dx.$$

$\downarrow$

$$2\pi i \operatorname{Res}\left[f(z), i\right]$$

$$= 2\pi i \left.\frac{e^{iz}}{2z}\right|_{z=i}$$

$$= 2\pi i \frac{e^{-1}}{2i}$$

$$= \frac{\pi}{e}$$

$\downarrow$

$$\int_C \frac{e^{i(R\cos\theta + iR\sin\theta)}\,R\,d\theta}{R^2 e^{i2\theta}+1}$$

$$\leq \int_C \frac{e^{-R\sin\theta}\,R\,d\theta}{|R^2 e^{i2\theta}+1|}$$

at $R \to \infty$, it tends to 0

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}$$