Definition :

$f(t)$ is defined for $t > 0$

Laplace transform of $f(t)$,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Q: 's' real or complex ?

↳ complex.

Q: Find the Laplace Transfrom of the following functions :-

(i) 1

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,1\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{1}{s}\left\{0 - 1\right\}$$

$$= \frac{1}{s}$$

(ii) t

$$\mathcal{L}[t] = \int_0^\infty e^{-st}\,t\,dt$$

$$= t\int e^{-st}dt\Big|_0^\infty - \int_0^\infty \left\{1\int e^{-st}dt\right\}dt$$

$$= t\left(-\frac{1}{s}\right)e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= +\frac{1}{s}\left(-\frac{1}{s}\right)\left[e^{-st}\right]_0^\infty$$

$$= \frac{1}{s^2}$$

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(iii) t^n

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

$$= t^n \underbrace{\int e^{-st}dt}_{\to 0}\Big|_0^\infty - n\int_0^\infty \left\{t^{n-1}\int \underline{\underline{e^{-st}}}\,dt\right\}dt$$

$$= -n\left(-\frac{1}{s}\right)\int_0^\infty t^{n-1}e^{-st}\,dt$$

$$= -n\left(-\frac{1}{s}\right)\left\{ t^{n-1}\underbrace{\int e^{-st}dt}_{\to 0}\Big|_0^\infty - (n-1)\int_0^\infty\left\{t^{n-2}\int e^{-st}dt\right\}dt\right.$$

$$= -n\left(-\frac{1}{s}\right)\{-(n-1)\}\left(-\frac{1}{s}\right)\int_0^\infty t^{n-2}e^{-st}dt$$

$$\cdots\cdots$$

$$= \frac{n(n-1)(n-2)\cdots\cdots 1}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$t^0 = \frac{0!}{s} = \frac{1}{s} \;;\; t^1 = \frac{1!}{s^2} \;;\; t^2 = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$t^3 = \frac{3!}{s^4} = \frac{6}{s^4}$$

(iv) $e^{\alpha t}$

$$\mathcal{L}\left[e^{\alpha t}\right] = \int_0^\infty e^{\alpha t} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-\alpha)t}\, dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-(s-\alpha)t}\, dt$$

$$= \lim_{p\to\infty} \left\{ \frac{1}{-(s-\alpha)}\, e^{-(s-\alpha)t} \Big|_0^p \right\}$$

$$= \lim_{p\to\infty} \left\{ \frac{1}{-(s-\alpha)} \cdot \left(\frac{1}{e^{(s-\alpha)p}} - 1 \right) \right\}$$

$$= \frac{1}{s-\alpha}$$

(v) $\sin(\alpha t)$

$$\mathcal{L}\left[\sin(\alpha t)\right] = \int_0^\infty \sin(\alpha t)\, e^{-st}\, dt$$

$$= \sin(\alpha t) \int e^{-st} dt \Big|_0^\infty - \alpha \int_0^\infty \left\{ \cos(\alpha t) \int e^{-st} dt \right\} dt$$

$$= \underbrace{\frac{\sin(\alpha t)}{-s}\, e^{-st} dt \Big|_0^\infty}_{\to 0} - \frac{\alpha}{(-s)} \int_0^\infty \cos(\alpha t)\, e^{-st}\, dt$$

$$= + \frac{\alpha}{s} \int_0^\infty \cos(\alpha t)\, e^{-st}\, dt$$

$$= \frac{\alpha}{s} \left[\cos(\alpha t) \int e^{-st} dt \Big|_0^\infty + \alpha \int_0^\infty \left\{ \sin(\alpha t) \int e^{-st} dt \right\} dt \right.$$

$$\mathcal{L}[\sin(\alpha t)] = \frac{\alpha}{s}\left[\cos(\alpha t)\int e^{-st}dt\Big|_0^\infty\right] + \frac{\alpha^2}{s(-s)}\int_0^\infty \sin(\alpha t)e^{-st}dt$$

$$= \frac{\alpha}{s}\left[\cos(\alpha t)\int e^{-st}dt\Big|_0^\infty\right] - \frac{\alpha^2}{s^2}\mathcal{L}[\sin(\alpha t)]$$

$$\Rightarrow \mathcal{L}[\sin(\alpha t)]\left(1+\frac{\alpha^2}{s^2}\right) = \frac{\alpha}{s}\lim_{p\to\infty}\left[\cos(\alpha t)\int_0^p e^{-st}dt\right]$$

$$\Rightarrow \mathcal{L}[\sin(\alpha t)] = \frac{s^2}{s^2+\alpha^2}\cdot\frac{\alpha}{s}\lim_{p\to\infty}\left\{\frac{\cos(\alpha t)}{(-s)}\left(e^{-pt}-1\right)\right\}$$

$$= \frac{s^2}{s^2+\alpha^2}\cdot\frac{\alpha}{s}\cdot\frac{1}{s}$$

$$= \frac{\alpha}{s^2+\alpha^2}$$

(vi) $\cos(\alpha t)$

$$\mathcal{L}[\cos(\alpha t)] = \int_0^\infty \cos(\alpha t)e^{-st}dt$$

$$= \cos(\alpha t)\int_0^\infty e^{-st}dt\Big|_0^\infty - \alpha\int_0^\infty\left\{-\sin(\alpha t)\int e^{-st}dt\right\}dt$$

$$= \cos(\alpha t)\frac{e^{-st}}{-s}\Big|_0^\infty - \frac{\alpha}{s}\int_0^\infty \sin(\alpha t)e^{-st}dt$$

$$= \left[0-\frac{1}{-s}\right] - \frac{\alpha}{s}\left\{\sin(\alpha t)\int e^{-st}dt\Big|_0^\infty - \alpha\int_0^\infty\left\{\cos(\alpha t)\int e^{-st}dt\right\}dt\right\}$$

$$= \frac{1}{s} - \frac{1}{s}\left\{ -\frac{\alpha^2}{(-s)} \int_0^\infty \cos(\alpha t)\, e^{-st}\, dt \right\}$$

$$\mathcal{L}[\cos(\alpha t)] = \frac{1}{s} - \frac{\alpha^2}{s^2}\, \mathcal{L}[\cos(\alpha t)]$$

$$\Rightarrow \mathcal{L}[\cos(\alpha t)] \left\{ 1 + \frac{\alpha^2}{s^2} \right\} = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}[\cos(\alpha t)] = \frac{1}{s} \cdot \frac{s^2}{s^2+\alpha^2}$$

$$= \frac{s}{s^2+\alpha^2}$$

Another method.

$$\mathcal{L}[e^{At}] = \frac{1}{s-A}$$

if $A = i\alpha$; then $\mathcal{L}[e^{i\alpha t}] = \frac{1}{s-i\alpha} = \frac{s+i\alpha}{s^2+\alpha^2} \longrightarrow (1)$

$$e^{i\alpha t} = \cos(\alpha t) + i \sin(\alpha t)$$

$$\mathcal{L}[e^{i\alpha t}] = \int_0^\infty e^{-st} \{ \cos(\alpha t) + i \sin(\alpha t) \}\, dt$$

$$= \int_0^\infty e^{-st} \cos(\alpha t)\, dt + i \int_0^\infty e^{-st} \sin(\alpha t)\, dt$$

$$= \mathcal{L}[\cos(\alpha t)] + i\, \mathcal{L}[\sin(\alpha t)] \rightarrow (2)$$

On equating real and imaginary parts.

$$\mathcal{L}[\cos(\alpha t)] = \frac{s}{s^2+\alpha^2} \; ; \; \mathcal{L}[\sin(\alpha t)] = \frac{\alpha}{s^2+\alpha^2}$$

(vii) $\mathcal{L}[\sinh(\alpha t)]$

$$\mathcal{L}[\sinh(\alpha t)] = \mathcal{L}\left[\frac{e^{\alpha t} - e^{-\alpha t}}{2}\right]$$

$$= \frac{1}{2}\left\{\mathcal{L}[e^{\alpha t}] - \mathcal{L}[e^{-\alpha t}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-\alpha} - \frac{1}{s+\alpha}\right\}$$

$$= \frac{1}{2}\,\frac{(s+\alpha)-(s-\alpha)}{s^2-\alpha^2}$$

$$= \frac{\alpha}{s^2-\alpha^2}$$

$$\mathcal{L}[\sin(\alpha t)] = \int_0^\infty \sinh(\alpha t)\, e^{-st}\, dt$$

$$= \underbrace{\sinh(\alpha t)\int e^{-st}\,dt\,\Big|_0^\infty}_{\to 0} - \alpha\int_0^\infty\left\{\cosh(\alpha t)\int e^{-st}\,dt\right\}dt$$

$$= -\frac{\alpha}{(-s)}\int_0^\infty \cosh(\alpha t)\, e^{-st}\, dt$$

$$= \frac{\alpha}{s}\left\{ \cosh(\alpha t)\frac{e^{-st}}{-s}\Bigg|_0^{\infty} - \frac{\alpha}{-s}\int_0^{\infty} \sinh(\alpha t)e^{-st}\,dt \right\}$$

$$\mathcal{L}[\sinh(\alpha t)] = \frac{\alpha}{s}\left\{ 0 - \left(\frac{1}{-s}\right) + \frac{\alpha}{s}\mathcal{L}[\sinh(\alpha t)] \right\}$$

$$\mathcal{L}[\sinh(\alpha t)]\left\{ 1 - \frac{\alpha^2}{s^2} \right\} = \frac{\alpha}{s^2}$$

$$\Rightarrow \mathcal{L}[\sinh(\alpha t)] = \frac{\alpha}{s^2}\left\{ \frac{s^2}{s^2-\alpha^2} \right\}$$

$$= \frac{\alpha}{s^2-\alpha^2}$$

viii. $\cosh(\alpha t)$

$$\mathcal{L}[\cosh(\alpha t)] = \mathcal{L}\left[\frac{e^{\alpha t}+e^{-\alpha t}}{2} \right]$$

$$= \frac{1}{2}\left\{ \mathcal{L}[e^{\alpha t}] + \mathcal{L}[e^{-\alpha t}] \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{s-\alpha} + \frac{1}{s+\alpha} \right\}$$

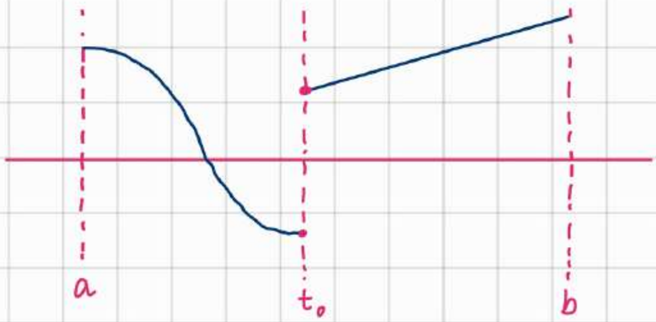
$$= \frac{1}{2}\,\frac{2s}{s^2-\alpha^2}$$

$$= \frac{s}{s^2-\alpha^2}$$

Intermission

1. Sectional or Piecewise Continuity

A function $f(t)$ can be called sectionally continuous in the interval $a \leq t \leq b$; if the interval $[a, b]$ can be subdivided into a finite no. of intervals in each of which the function $f(t)$ is continuous and has finite LHL and RHL.



Notation :

RHL at t_0 : $\lim\limits_{\epsilon \to 0} f(t_0 + \epsilon) \equiv F(t_0 +)$

LHL at t_0 = $\lim\limits_{\epsilon \to 0} f(t_0 - \epsilon) \equiv F(t_0 -)$

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2. Functions of Exponential Order.

$f(t)$ is function of exponential order if

real constants $M > 0$ and γ exist

such that for all $t > N$

$$\left| e^{-\gamma t} f(t) \right| < M \quad \text{or} \quad |f(t)| < \underline{\underline{Me^{\gamma t}}}$$

Eg. t^2

$\to |t|^2 = t^2$;

1. e^{3t}

for all $t > 0$; $t^2 < e^{3t}$

So, t^2 is of exponential order for $t > 0$.

Eg. e^{t^3}

$$\left| e^{-\gamma t} e^{t^3} \right| = e^{t^3 - \gamma t} < M$$

$\gamma, M \to \times$; e^{t^3} is not of exp. order.

** $\sin(\alpha t)$; $\cos(\alpha t)$ are of exponential order.

Q: When does Laplace Transform exist ?

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Theorem 1.1 — If $f(t)$ is sectionally continuous in every finite interval between $0 \leq t \leq N$ and $f(t)$ is of exponential order γ for $t < N$ then its Laplace transform exist for all $s > \gamma$.

Note : If this condition is not satisfied, then Laplace Transform of $f(t)$ may or may not exist.