PROPERTIES OF LAPLACE TRANSFORM.

A. Linearity Property.

Theorem 1.2 - For constants c_1 and c_2,

$$\mathcal{L}\left[c_1 f_1(t) + c_2 f_2(t)\right] = c_1 \mathcal{L}\left[f_1(t)\right] + c_2 \mathcal{L}\left[f_2(t)\right]$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

* $\mathcal{L}$ may be called the Laplace transform operator
* $\mathcal{L}$ is a linear operator as it preserves the linearity property.

Proof : $\mathcal{L}\left[c_1 f_1(t) + c_2 f_2(t)\right]$

$$= \int_0^\infty c_1 f_1(t) e^{-st} dt + \int_0^\infty c_2 f_2(t) e^{-st} dt$$

$$= c_1 \int_0^\infty f_1(t) e^{-st} dt + c_2 \int_0^\infty f_2(t) e^{-st} dt$$

$$= c_1 \mathcal{L}\left[f_1(t)\right] + c_2 \mathcal{L}\left[f_2(t)\right]$$

Example : Using the linearity property find -

$$\mathcal{L}\left[4e^{5t} + 6t^3 - 3\sin(4t) + 2\cos(2t)\right]$$

$$= 4\mathcal{L}\left[e^{5t}\right] + 6\mathcal{L}\left[t^3\right] - 3\mathcal{L}\left[\sin(4t)\right] + 2\mathcal{L}\left[\cos(2t)\right]$$

$$= \frac{4}{s-5} + 6\cdot\frac{3!}{s^4} - 3\cdot\frac{4}{s^2+16} + 2\frac{s}{s^2+4}$$

$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$$

B. First Translation or Shifting Property

Theorem 1.3 - If $\mathcal{L}\left[f(t)\right] = F(s)$

then $\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$

For eg, $\mathcal{L}\left[\cos(2t)\right] = \dfrac{s}{s^2+4}$

then we have $\mathcal{L}\left[e^{-t}\cos(2t)\right]$

$$= \frac{s-(-1)}{(s-(-1))^2+2^2} = \frac{s+1}{s^2+2s+5}$$

Proof : $\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}\,dt = F(s)$

then $\mathcal{L}\left[e^{\alpha t}f(t)\right] = \int_0^\infty f(t)\,e^{\alpha t}e^{-st}\,dt$

$$= \int_0^\infty f(t)\,e^{-(s-\alpha)t}\,dt$$

$$= F(s-a)$$

Examples : Using the above property, find the Laplace Transform of the following functions :

(i) $\mathcal{L}\left[t^2e^{3t}\right]$

We have, $\mathcal{L}\left[t^2\right] = \frac{2}{s^3}$

Hence, $\mathcal{L}\left[t^2e^{3t}\right] = \frac{2}{(s-3)^3}$

(ii) $\mathcal{L}\left[e^{-2t}\sin(4t)\right]$

We have, $\mathcal{L}\left[\sin(4t)\right] = \frac{4}{s^2+16}$

Hence, $\mathcal{L}\left[e^{-2t}\sin(4t)\right]$

$$= \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$$

(iii) $\mathcal{L}\left[e^{4t}\cosh(5t)\right]$

$\mathcal{L}\left[\cosh(5t)\right] = \frac{s}{s^2-25}$

Hence, $\mathcal{L}\left[e^{4t}\cosh(5t)\right] = \frac{s-4}{(s-4)^2-25} = \frac{s-4}{s^2-8s-9}$

C. Second Translation or Shifting Property.

Theorem 1.4 – If $\mathcal{L}[f(t)] = F(s)$

and $g(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$, then

$$\mathcal{L}[g(t)] = e^{-as} F(s).$$

For eg, $\mathcal{L}[t^3] = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$

So, Laplace transform of $g(t) = \begin{cases} (t-2)^3 & \text{for } t>2 \\ 0 & \text{for } t<0 \end{cases}$

is $\mathcal{L}[g(t)] = e^{-2s} \dfrac{6}{s^4} = \dfrac{6e^{-2s}}{s^4}$.

Proof :- $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$\mathcal{L}[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} . 0 . dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

Let, $t = u + a$ | as $t \to a$; $u \to 0$

$\Rightarrow dt = du$ | as $t \to \infty$; $u \to \infty$

$$\therefore \mathcal{L}[g(t)] = \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-as} F(s)$$

Example : Find $\mathcal{L}[f(t)]$ if $f(t) = \begin{cases} \cos(t - 2\pi/3) & \text{for } t > \frac{2\pi}{3} \\ 0 & \text{for } t < \frac{2\pi}{3} \end{cases}$

$$\mathcal{L}[f(t)] = e^{-\frac{2\pi s}{3}} \mathcal{L}[\cos(t)]$$

$$= e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1} = \frac{s e^{-\frac{2\pi s}{3}}}{s^2+1}$$

D. Change of Scale Property.

Theorem 1.5 - If $\mathcal{L}[f(t)] = F(s)$

then $\mathcal{L}[f(at)] = \frac{1}{a} f\left(s/a\right)$

For eg, $\mathcal{L}[\sin(t)] = \frac{1}{s^2+1}$

$$\mathcal{L}[\sin(3t)] = \frac{1}{3} \cdot \frac{1}{(s/3)^2+1} = \frac{1}{3} \frac{3^2}{s^2+9} = \frac{3}{s^2+9}$$

Proof :- $\mathcal{L}[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$

Let, $at = u$ $\Rightarrow dt = \frac{1}{a} du$ | As $t \to 0$; $u \to 0$; $t \to \infty$; $u \to \infty$

$$\therefore \mathcal{L}[f(at)] = \int_0^\infty e^{-s\left(\frac{u}{a}\right)} f(u) \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}u} f(u)\, du$$

$$= \frac{1}{a} F\left(s/a\right).$$

Example : Given that $\mathcal{L}\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$ then find $\mathcal{L}\left[\frac{\sin(at)}{t}\right]$.

$$\mathcal{L}\left[\frac{\sin(at)}{at}\right] = \frac{1}{a} \tan^{-1}\left(\frac{1}{s/a}\right) = \frac{1}{a} \tan^{-1}\left(\frac{a}{s}\right)$$

$$\Rightarrow \frac{1}{a} \mathcal{L}\left[\frac{\sin(at)}{t}\right] = \frac{1}{a} \tan^{-1}\left(a/s\right)$$

$$\Rightarrow \mathcal{L}\left[\frac{\sin(at)}{t}\right] = \tan^{-1}\left(a/s\right)$$

E. Laplace Transform of Derivatives

Theorem 1.6 – If $\mathcal{L}[f(t)] = F(s)$

then $\mathcal{L}[f'(t)] = sF(s) - f(0)$

if $f(t)$ is continuous for the interval $[0, N]$ and of exponential order for $t > N$ while $f'(t)$ is sectionally / piecewise continuous for $[0, N]$

For eg: $f(t) = \cos(3t)$; $\mathcal{L}[\cos(3t)] = \dfrac{s}{s^2+9} = F(s)$

$$\mathcal{L}[f'(t)] = s\frac{s}{s^2+9} - \cos(0)$$

$$= \frac{s^2}{s^2+9} - 1 = \frac{s^2 - s^2 - 9}{s^2+9} = -\frac{9}{s^2+9}$$

Proof :-

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st}\int f'(t)\,dt \Big|_0^\infty - \int_0^\infty \left\{ \left(-se^{-st}\right) \int f'(t)\,dt \right\} dt$$

$$= e^{-st} f(t) \Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sF(s)$$

$$= sF(s) - f(0)$$

Theorem 1.7 – If in theorem 1.6, $f(t)$ fails to be continuous at $t=0$, but $\lim_{t\to 0^+} f(t) = f(0+)$ exists, (but is not equal to $f(0)$, which may or may not exist)

then $$\mathcal{L}[f'(t)] = sF(s) - f(0+)$$

$$\mathcal{L}[f'(t)] = sF(s) - \lim_{t\to 0^+} f(t)$$

Theorem 1.8 – If in Theorem 1.6, $f(t)$ fails to be continuous at $t=a$, then

$$\mathcal{L}[f'(t)] = sF(s) - f(0) - e^{-as}\{f(a+) - f(a-)\}$$

$$= sF(s) - f(0) - e^{-as}\underbrace{\left\{\lim_{t\to a^+} f(t) - \lim_{t\to a^-} f(t)\right\}}_{\text{called jump at discontinuity } t=a}$$

* For more than one discontinuity. appropriate modifications may be made.

Theorem 1.9 – If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

provided both $f'(t)$ and $f(t)$ are continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ while $f''(t)$ is sectionally continuous for $0 \leq t \leq N$.

* For discontinuities, appropriate modifications may be made.

Theorem 1.10 - Laplace transform of n^{th} order

If $\mathcal{L}[f(t)] = F(s)$ then,

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - s^2 f^{(n-3)}(0) - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$\hookrightarrow$ $\mathcal{L}[f^{(2)}(t)] = s^2 F(s) - s f(0) - f'(0)$

$\hookrightarrow$ $\mathcal{L}[f^{(3)}(t)] = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$

* this holds if $f(t), f'(t) \dots, f^{(n-1)}(t)$ are continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ while $f^{(n)}(t)$ is sectionally continuous for $0 \leq t \leq N$

Q: Prove theorem 1.9. i.e. $\mathcal{L}[f''(t)] = s^2 F(s) - s f(0) - f'(0)$

Let $g(t) = f'(t)$

$$\mathcal{L}[g'(t)] = s G(s) - g(0)$$

$$\Rightarrow \mathcal{L}[f''(t)] = s \mathcal{L}[f'(t)] - f'(0)$$

$$= s \{ s F(s) - f(0) \} - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

Q: Derive the Laplace Transforms using 1.6.

a) $\mathcal{L}[1] = \frac{1}{s}$

Let, $f(t) = 1$

$f'(t) = 0$

$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$

$\Rightarrow \mathcal{L}[0] = s\mathcal{L}[1] - 1$

$\Rightarrow 0 = s\mathcal{L}[1] - 1$

$\Rightarrow \mathcal{L}[1] = \frac{1}{s}$.

b) $\mathcal{L}[t] = \frac{1}{s^2}$

Let, $f(t) = t$

$f'(t) = 1$

$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$

$\Rightarrow \mathcal{L}[1] = s\mathcal{L}[t] - 0$

$\Rightarrow \frac{1}{s} = s\mathcal{L}[t]$

$\Rightarrow \mathcal{L}[t] = \frac{1}{s^2}$

c) $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

Let, $f(t) = e^{at}$

$\Rightarrow f'(t) = a e^{at}$

$= a f(t)$

$$\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)$$

$$\Rightarrow \alpha\,\mathcal{L}[f(t)] = s\,\mathcal{L}[f(t)] - e^{0}$$

$$\Rightarrow \mathcal{L}[f(t)]\{s-\alpha\} = 1$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{1}{s-\alpha}$$

d) $\mathcal{L}[\sin(\alpha t)] = \dfrac{\alpha}{s^2+\alpha^2}$

Let, $f(t) = \sin(\alpha t)$

$f'(t) = \alpha \cos(\alpha t)$

$f''(t) = -\alpha^2 \sin(\alpha t)$

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - s f(0) - f'(0)$$

$$\Rightarrow -\alpha^2\,\mathcal{L}[\sin(\alpha t)] = s^2\mathcal{L}[\sin(\alpha t)] - s\cdot\sin(0) - \alpha\cdot\cos(0)$$

$$\Rightarrow \mathcal{L}[\sin(\alpha t)]\{s^2+\alpha^2\} = +\alpha$$

$$\Rightarrow \mathcal{L}[\sin(\alpha t)] = \frac{\alpha}{s^2+\alpha^2}$$