Properties of Laplace Transform

F. Laplace Transform of Integrals

Theorem 1.11 - If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$$

Example : if $\mathcal{L}[\sin(2t)] = \dfrac{2}{s^2+4}$, then $\mathcal{L}\left[\int_0^t \sin(2u)\,du\right] = ?$

$$\int_0^t \sin(2u)\,du = -\frac{\cos(2u)}{2}\Big|_0^t = -\frac{1}{2}\left\{\cos(2t) - 1\right\}$$

$$\mathcal{L}\left[-\frac{1}{2}\left\{\cos(2t) - 1\right\}\right] = -\frac{1}{2}\left[\frac{s}{s^2+4} - \frac{1}{s}\right]$$

$$= -\frac{1}{2}\left[\frac{s^2 - (s^2+4)}{s(s^2+4)}\right]$$

$$= \frac{2}{s(s^2+4)}$$

$$= \frac{1}{s}\,\mathcal{L}\left[\sin(2t)\right]$$

$$= \frac{1}{s}F(s)$$

Proof :- Let $g(t) = \int_0^t f(u)\,du$

then $g'(t) = f(t)$ and $g(0) = 0$

Taking Laplace Transform on both sides,

$$\mathcal{L}[g'(t)] = s\,\mathcal{L}[g(t)] - g(0)$$

$$\Rightarrow \mathcal{L}[f(t)] = s\,\mathcal{L}\left[\int_0^t f(u)\,du\right] - 0$$

$$\Rightarrow F(s) = s\,\mathcal{L}\left[\int_0^t f(u)\,du\right]$$

$$\Rightarrow \mathcal{L}\left[\int_0^t f(u)\,du\right] = \frac{1}{s}F(s)$$

G. Multiplication by t^n

Theorem 1.12 - If $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

Example : Since $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$

$\therefore \mathcal{L}[te^{2t}] = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$

$\mathcal{L}[t^2e^{2t}] = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = -\frac{d}{ds}\left(\frac{1}{(s-2)^2}\right) = \frac{2}{(s-2)^3}$

also obvious from the translation / shifting property.

Proof :

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}(t\,f(t))\,dt \longrightarrow (1)$$

$$\frac{d^2F}{ds^2} = (-1)^2 \int_0^\infty e^{-st}\{t^2 f(t)\}\,dt$$

To establish this theorem in general, we use mathematical induction. Assume that the theorem is true for $n=k$. i.e. assume

$$\int_0^\infty e^{-st}\{t^k f(t)\}\,dt = (-1)^k F^{(k)}(s) \longrightarrow ②$$

Then, $$\frac{d}{ds}\int_0^\infty e^{-st}\{t^k f(t)\}\,dt = (-1)^k F^{(k)}(s)$$

$$\Rightarrow -\int_0^\infty e^{-st}\{t^{k+1} f(t)\}\,dt = (-1)^k F^{(k+1)}(s)$$

$$\Rightarrow \int_0^\infty e^{-st}\{t^{k+1} f(t)\}\,dt = (-1)^{k+1} F^{(k+1)}(s) \longrightarrow ③$$

If ② is true i.e. if the theorem holds for $n=k$, then ③ is true.

For any $n=k+1$ this theorem holds.

By ①, the theorem is true for $n=1$.

Hence, it is true for all $\mathbb{N}$.

Example: $\mathcal{L}[t\sin(at)] = ?$

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$$

Hence, $\mathcal{L}[t\sin(at)]$

$$= (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

$$\mathcal{L}\left[t^2\cos(at)\right] = ?$$

$$\mathcal{L}\left[\cos(at)\right] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\left[t^2\cos(at)\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{s^2+a^2-s.2s}{(s^2+a^2)^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{a^2-s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{(s^2+a^2)^2(-2s)-(a^2-s^2)2(s^2+a^2)2s}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2)-(a^2-s^2)4s}{(s^2+a^2)^3}$$

$$= \frac{-2s^3-2sa^2-4sa^2+4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3-6a^2s}{(s^2+a^2)^3}$$

H. Division by t

Theorem 1.13 - If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,du \quad \text{provided } \lim_{t\to 0}\frac{f(t)}{t} \text{ exists.}$$

Example : $\mathcal{L}\left[\frac{\sin(t)}{t}\right] = ?$

$$\mathcal{L}[\sin(t)] = \frac{1}{s^2+1} \quad \text{also, } \lim_{t\to 0}\left(\frac{\sin t}{t}\right) = 1.$$

Hence, $\mathcal{L}\left[\frac{\sin(t)}{t}\right] = \lim_{p\to\infty}\int_s^p \frac{du}{u^2+1}$

$$= \lim_{p\to\infty} \tan^{-1}(u)\Big|_s^p$$

$$= \lim_{p\to\infty}\left\{\tan^{-1}(p) - \tan^{-1}(s)\right\}$$

$$= \lim_{p\to\infty} \tan^{-1}\left(\frac{p-s}{1+ps}\right)$$

$$= \lim_{p\to\infty} \tan^{-1}\left\{\frac{p(1-s/p)}{p(\frac{1}{p}+s)}\right\}$$

$$= \lim_{p\to\infty} \tan^{-1}\left\{\frac{1-s/p}{\frac{1}{p}+s}\right\}$$

$$= \tan^{-1}\left(\frac{1}{s}\right).$$

Proof :- Let $g(t) = \dfrac{f(t)}{t}$

then $f(t) = t\,g(t)$

$$\mathcal{L}[f(t)] = -\frac{d}{ds}\mathcal{L}[g(t)]$$

$$\Rightarrow F(s) = -\frac{d}{ds}G(s)$$

Integrating $G(s) = -\displaystyle\int_{\infty}^{s} F(u)\,du$ | choosing $\lim_{s\to\infty} G(s) = 0$

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_{s}^{\infty} F(u)\,du$$

Example: find $\mathcal{L}\left[\displaystyle\int_0^t \frac{\sin(u)}{u}\,du\right]$

$$\mathcal{L}\left[\frac{\sin(t)}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$$

Hence $\mathcal{L}\left[\displaystyle\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

Q: Prove that $\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(u)\,du$ provided that the integrals converge.

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,du$$

$$\Rightarrow \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \int_s^\infty F(u)\,du$$

$$\lim_{s\to 0^+}\int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \lim_{s\to 0^+}\int_s^\infty F(u)\,du$$

$$\Rightarrow \int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(u)\,du$$

provided that the integrals converge for $s \to 0^+$.

Q: Show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Let, $f(t) = \sin(t)$; $\mathcal{L}[f(t)] = \frac{1}{s^2+1}$

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,du$$

$$\Rightarrow \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \lim_{s\to 0}\int_s^\infty \frac{1}{1+u^2}\,du$$

$$\Rightarrow \int_0^\infty \frac{\sin(t)}{t}\, dt = \lim_{s\to 0} \tan^{-1}(u)\Big|_s^\infty$$

$$= \lim_{s\to 0}\left(\tan^{-1}(\infty) - \tan^{-1}(s)\right)$$

$$= \pi/2 - 0 = \pi/2 .$$

1. Periodic Functions.

Theorem 1.14 — Let $f(t)$ have a period T $(T>0)$ such that

$$f(t+T) = f(t)$$

then

$$\mathcal{L}\left[f(t)\right] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}} = F(s)$$

Proof :

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \dots\dots$$

In the second integral let $t = u + T$
In the third integral let $t = u + 2T$

$$\mathcal{L}\left[f(t)\right] = \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-s(u+T)} f(u+T)\,du +$$

$$\int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-s(u+T)} f(u+T)\,du +$$

$$\int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right) \int_0^T e^{-su} f(u)\,du$$

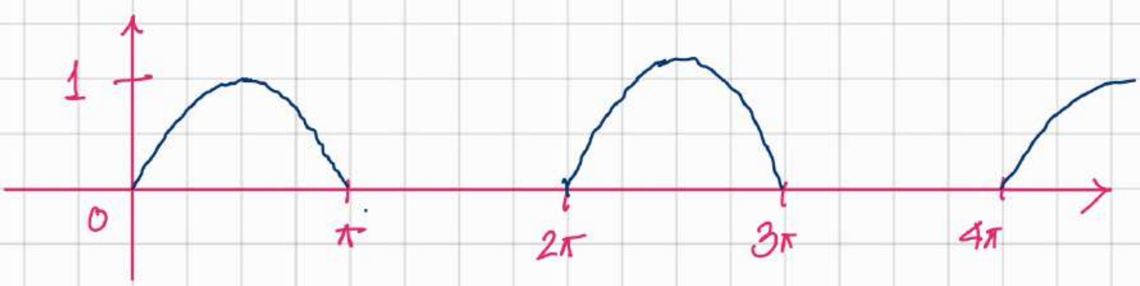
$$= \frac{\int_0^T e^{-su} f(u)\,du}{1 - e^{-sT}}$$

$$= \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

Example: a) Graph the function $f(t) = \begin{cases} \sin(t) & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$

extended periodically with a period 2π.

b) Find $\mathcal{L}[f(t)]$



$T \to 2\pi$

So. $\mathcal{L}[f(t)]$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\sin(t) \int e^{-st} \Big|_0^{\pi} - \int_0^{\pi} \Big(\cos t \underbrace{\int e^{-st} dt}_{-\frac{1}{s}} \Big) dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\cancel{\frac{\sin(t)\, e^{-st}}{-s}}^{\,0} \Big|_0^{\pi} + \frac{1}{s} \int_0^{\pi} \cos t\, e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \frac{1}{s} \left[\cos t \int e^{-st} dt \Big|_0^{\pi} - \frac{1}{s} \int_0^{\pi} \sin(t)\, e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \cdot \frac{1}{s} \left[\frac{\cos t\; e^{-st}}{-s} \Big|_0^{\pi} - \frac{1}{s} \left(1-e^{-2\pi s}\right) \mathcal{L}(f(t)) \right]$$

$$\mathcal{L}[f(t)] \left\{ 1 + \frac{1}{s^2} \right\} = \frac{1}{1-e^{-2\pi s}} \cdot \frac{1}{(-s^2)} \left(-e^{-s\pi} - 1 \right)$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2\pi s}} \cdot \left[\frac{1+e^{-\pi s}}{1+s^2} \right]$$

J. Behaviour of $F(s)$ as $s \to \infty$

Theorem 1.15 → If $\mathcal{L}[f(t)] = F(s)$

then $\lim\limits_{s\to\infty} F(s) = 0$

Initial Value Theorem

Theorem 1.16 → If the indicated limit exists, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Final Value Theorem

Theorem 1.17 → If the indicated limit exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

> Note:
>
> If $\lim\limits_{t\to 0} \dfrac{f(t)}{g(t)} = 1$; we say $f(t)$ is close to $g(t)$
>
> and we write $f(t) \sim g(t)$ as $t \to 0$

Generalisation of Initial Value theorem

Theorem 1.18 – If $f(t) \sim g(t)$ as $t \to 0$,

then $F(s) \sim G(s)$ as $s \to \infty$

Generalisation of final value theorem

Theorem 1.19 – If $f(t) \sim g(t)$ as $t \to \infty$

then $F(s) \sim G(s)$ as $s \to 0$.

K. Methods of finding Laplace Transform.

1. Direct Method : $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

2. Series Method :

If $f(t)$ has a power series expansion given by

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots = \sum_{n=0}^{\infty} a_n t^n$$

then Laplace Transform of $f(t)$ can be obtained by taking the sum of the Laplace Transforms of each term of the series.

$$\mathcal{L}[f(t)] = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!\,a_2}{s^3} + \frac{3!\,a_3}{s^4} + \dots = \sum_{n=0}^{\infty} \frac{n!\,a_n}{s^{n+1}}$$

* vaid if series is convergent for $s > \gamma$.

3. Other methods ....