Definition.

If the Laplace transform of a function $f(t)$ is $F(s)$.

i.e. $\mathcal{L}[f(t)] = F(s)$

then $f(t)$ is called the Inverse Laplace Transform of $F(s)$.

i.e. $f(t) = \mathcal{L}^{-1}[F(s)]$

$\mathcal{L}^{-1}$ is called the Inverse Laplace Transformation operator.

Eg: since $\mathcal{L}[e^{-3t}] = \dfrac{1}{s+3}$

so $\mathcal{L}^{-1}\left[\dfrac{1}{s+3}\right] = e^{-3t}$

Uniqueness of Inverse Laplace Transform

Theorem 2.1 (Lerch's Theorem): If we restrict ourselves to functions $f(t)$ which are sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order for $t > N$, then the inverse Laplace transform of $f(s)$ ie.

$$\mathcal{L}^{-1}[F(s)] = f(t) \text{ is unique.}$$

We shall always assume such uniqueness unless otherwise stated.

$$f_1(t) = e^{-3t} \; ; \; f_2(t) = \begin{cases} 0 & t=1 \\ e^{-3t} & \text{otherwise} \end{cases}$$

both have the same Laplace transform $\frac{1}{s+3}$.

If we allow null functions, we see that the inverse Laplace transform is not unique.

It is unique, if we disallow null functions.

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1. Find the following inverse Laplace transforms :-

a) $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right]$ Since. $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \therefore \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

b) $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]$ $\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \therefore \mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{1}{n!}t^n$

c) $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right]$ $\mathcal{L}\left[\sinh(at)\right] = \frac{a}{s^2-a^2} \therefore \mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh(at)}{a}$

d) $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]$ $\mathcal{L}\left[\cos(at)\right] = \frac{s}{s^2+a^2} \therefore \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$

Q: Prove that $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$ for $n > -1$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st} t^n \, dt$$

Let, $st = x$, for $s > 0$

$$\therefore \mathcal{L}[t^n] = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n d\left(\frac{x}{s}\right)$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} \, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\Rightarrow \mathcal{L}\left[\frac{t^n}{\Gamma(n+1)}\right] = \frac{1}{s^{n+1}}$$

Hence, $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$.

Q. Find $\mathcal{L}^{-1}\left[\frac{1}{s^{3/2}}\right]$

using the result above.

$$\mathcal{L}^{-1}\left[\frac{1}{s^{3/2}}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{\frac{1}{2}+1}}\right] = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{2t^{1/2}}{\sqrt{\pi}}$$

Properties of Laplace Transform

A. Linearity Property

Theorem 2.2 – If c_1 and c_2 are any constants while $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$\mathcal{L}^{-1}\left[c_1F_1(s) + c_2F_2(s)\right] = c_1\mathcal{L}^{-1}\left[F_1(s)\right] + c_2\mathcal{L}^{-1}\left[F_2(s)\right]$$

$$= c_1 f_1(t) + c_2 f_2(t)$$

* $\mathcal{L}^{-1}$ is a linear operator

Example : $\mathcal{L}^{-1}\left[\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right]$

$$= 4\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - 3\mathcal{L}^{-1}\left[\frac{s}{s^2+16}\right] + \frac{5}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= 4e^{2t} - 3\cos(4t) + \frac{5}{2}\sin(2t)$$

Example : $\mathcal{L}^{-1}\left[\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right]$

$$= 5\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{s^3}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right]$$

$$+ 6\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] + 4\mathcal{L}^{-1}\left[\frac{6}{s^4}\right] - 30\mathcal{L}^{-1}\left[\frac{1}{s^{7/2}}\right]$$

$$= 5t + 2t^2 - 2\cos(3t) + 6\sin(3t) + 4t^3 - 30\frac{t^{5/2}}{\Gamma(7/2)}$$

B. First Translation or Shifting Property

Theorem 2.3 – If $\mathcal{L}^{-1}[F(s)] = f(t)$

then $\mathcal{L}^{-1}[F(s-b)] = e^{bt} f(t)$

Example: $\mathcal{L}^{-1}\left[\dfrac{1}{s^2-2s+5}\right] = ?$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+5}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+4}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-1)^2+4}\right]$$

Since, $\mathcal{L}[\sin(2t)] = \dfrac{2}{s^2+4}$

Hence. $\mathcal{L}^{-1}\left[\dfrac{1}{s^2-2s+5}\right] = \dfrac{1}{2}\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)^2+4}\right]$

$$= \frac{1}{2} e^{t} \sin(2t)$$

Proof : We previously proved theorem 1.3 -

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$$

$$\therefore \mathcal{L}^{-1}\left[F(s-b)\right] = e^{bt}f(t)$$

Another method : Since $F(s) = \int_0^\infty e^{-st}f(t)\,dt$

$$\therefore F(s-a) = \int_0^\infty e^{-(s-a)t}f(t)\,dt = \int_0^\infty e^{-st}\left(e^{at}f(t)\right)dt$$

$$= \mathcal{L}\left[e^{at}f(t)\right]$$

$$\therefore \mathcal{L}^{-1}\left[F(s-a)\right] = e^{at}f(t)$$

Q: Using the first translation property find the inverse Laplace Transform of the following :-

a) $$\frac{6s-4}{s^2-4s+20} = \frac{6s-4}{s^2-4s+4+16} = \frac{6s-4}{(s-2)^2+4^2}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{6s-4}{(s-2)^2+4^2}\right] = \mathcal{L}^{-1}\left[\frac{6(s-2)}{(s-2)^2+4^2}\right] + \mathcal{L}^{-1}\left[\frac{2\cdot 4}{(s-2)^2+4^2}\right]$$

$$= 6e^{2t}\cos(4t) + 2e^{2t}\sin(4t)$$

$$= 2e^{2t}\left[3\cos(4t) + \sin(4t)\right]$$

b) $\frac{4s+12}{s^2+8s+16} = \frac{4(s+4)-4}{(s+4)^2} = 4\left(\frac{1}{s+4}\right) - 4\left(\frac{1}{(s+4)^2}\right)$

$\therefore \mathcal{L}^{-1}\left[\frac{4s+12}{s^2+8s+16}\right] = 4\mathcal{L}^{-1}\left[\frac{1}{s+4}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{(s+4)^2}\right]$

$= 4e^{-4t}.1 - 4e^{-4t}.t$

$= 4e^{-4t}(1-t)$

c) $\frac{3s+7}{s^2-2s-3} = \frac{3(s-1)+10}{(s-1)^2-4} = 3\frac{(s-1)}{(s-1)^2-2^2} + 5\frac{2}{(s-1)^2-2^2}$

$\therefore \mathcal{L}^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = 3\mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2-2^2}\right] + 5\mathcal{L}^{-1}\left[\frac{2}{(s-1)^2-2^2}\right]$

$= 3e^t\cosh(2t) + 5e^t\sinh(2t)$

d) $\frac{1}{\sqrt{2s+3}} = \frac{1}{\sqrt{2}}\left(\frac{1}{(s+3/2)^{1/2}}\right)$

$\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{t^{n+1}}$

$\therefore \mathcal{L}^{-1}\left[\frac{1}{\sqrt{2s+3}}\right] = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left[\frac{1}{(s+3/2)^{1/2}}\right] = \frac{1}{\sqrt{2}}e^{-3t/2}\frac{t^{-1/2}}{\Gamma(1/2)}$

$= \frac{1}{\sqrt{2\pi}}t^{-1/2}e^{-3/2}$

C. Second Translation or Shifting Property.

Theorem 2.4 — If $\mathcal{L}^{-1}[F(s)] = f(t)$

then $\mathcal{L}^{-1}[e^{-bs}F(s)] = \begin{cases} f(t-b) & t > b \\ 0 & t < b \end{cases}$

Example : $\mathcal{L}^{-1}\left[\frac{e^{-\pi s/3}}{s^2+1}\right] = ?$

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s/3}}{s^2+1}\right]$$

$$= \mathcal{L}^{-1}\left[e^{-\frac{\pi}{3}s} \cdot \frac{1}{s^2+1}\right]$$

Since, $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin(t)$

Hence, $\mathcal{L}^{-1}\left[e^{-\pi/3 s} \cdot \frac{1}{s^2+1}\right] = \begin{cases} \sin(t-\pi/3) & \text{for } t > \pi/3 \\ 0 & \text{for } t > 0 \end{cases}$

Proof :- Since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore \quad e^{-bs}F(s) = \int_0^\infty e^{-bs} e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-s(t+b)} f(t)\,dt$$

Let $t+b = u$

as $t \to 0$; $u \to b$
as $t \to \infty$; $u \to \infty$

$$\therefore \quad e^{-bs} F(s) = \int_b^{\infty} e^{-su} f(u-b)\, du$$

$$= \int_0^b e^{-su} . 0 . du + \int_b^{\infty} e^{-su} f(u-b) dt$$

$$\therefore \mathcal{L}^{-1}\left[e^{-bs} F(s)\right] = \begin{cases} 0 & \text{if } t < b \\ f(u-b) & \text{if } t > b \end{cases}$$

Q Using the second translation property find the inverse Laplace transform of each of the following :-

a) $\dfrac{e^{-5s}}{(s-2)^4}$ $\qquad \mathcal{L}^{-1}\left[\dfrac{1}{(s-2)^4}\right] = e^{2t} \dfrac{t^3}{3!} = \dfrac{1}{6} t^3 e^{2t}$

$$\therefore \mathcal{L}^{-1}\left[\frac{e^{-5s}}{(s-2)^4}\right] = \begin{cases} \frac{1}{6}(t-b)^3 e^{2(t-5)} & \text{for } t > 5 \\ 0 & \text{for } t < 5 \end{cases}$$

b) $\dfrac{se^{-\frac{4\pi s}{5}}}{s^2+25}$ $\qquad \mathcal{L}^{-1}\left[\dfrac{s}{s^2+25}\right] = \cos(5t)$

$$\therefore \mathcal{L}^{-1}\left[\frac{se^{-\frac{4\pi s}{5}}}{s^2+25}\right] = \begin{cases} \cos(5(t-4\pi/5)), & t > 4\pi/5 \\ 0, & t < 4\pi/5 \end{cases}$$

$$= \begin{cases} \cos(5t-4\pi), & t > 4\pi/5 \\ 0, & t < 4\pi/5 \end{cases}$$

$$= \cos(5t)\,\mathcal{U}(t-4\pi/5)$$

c) $\dfrac{(s+1)e^{-\pi s}}{s^2+s+1}$ $\qquad \mathcal{L}^{-1}\left[\dfrac{s+1}{s^2+2\cdot s\cdot\frac{1}{2}+\frac{1}{4}+\frac{3}{4}}\right]$

$$= \mathcal{L}^{-1}\left[\frac{s+\frac{1}{2}+\frac{3}{4}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]$$

$$= e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + e^{-\frac{1}{2}t}\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$= \frac{e^{-t/2}}{\sqrt{3}}\left\{\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) + \sin\left(\frac{\sqrt{3}}{2}t\right)\right\}$$

Thus, $\mathcal{L}^{-1}\left[\dfrac{(s+1)\,e^{-\pi s}}{s^2+s+1}\right]$

$$= \begin{cases} \dfrac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}} \left\{ \sqrt{3}\cos\left(\frac{\sqrt{3}}{2}(t-\pi)\right) + \sin\left(\frac{\sqrt{3}}{2}(t-\pi)\right) \right\} & \text{for } t>\pi \\ 0 \quad \text{for } t<\pi \end{cases}$$

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d) $\mathcal{L}^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right] = \mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{\Gamma(5/2)} \cdot e^{4} \dfrac{\Gamma(5/2)}{(s+4)^{5/2}}\right]$

Since, $\mathcal{L}^{-1}\left[\dfrac{\Gamma(5/2)}{(s+4)^{5/2}}\right] = e^{-4t}t^{3/2}$

$$\therefore \mathcal{L}^{-1}\left[\frac{e^{4-3s}}{(s+4)^{5/2}}\right] = \begin{cases} \dfrac{e^4}{\Gamma(5/2)} \cdot e^{-4(t-3)}(t-3)^{3/2} & \text{for } t>3 \\ 0 \quad \text{for } t<3 \end{cases}$$

$$= \begin{cases} \dfrac{4}{3\sqrt{\pi}} \cdot e^{-4(t-4)}(t-3)^{3/2} & \text{for } t>3 \\ 0 \quad \text{for } t<3 \end{cases}$$

D. Change of Scale Property

Theorem 2.5 – If $\mathcal{L}^{-1}[F(s)] = f(t)$

then $\mathcal{L}^{-1}[F(bs)] = \frac{1}{b} f(t/b)$

Example : $\mathcal{L}^{-1}\left[\frac{2s}{(2s)^2+16}\right] = \frac{1}{2}\cos\left(\frac{4t}{2}\right) = \frac{1}{2}\cos(2t)$

Proof :-

$$\text{Since } F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \quad F(ks) = \int_0^\infty e^{-kst} f(t)\,dt \qquad (\text{for } k > 0)$$

Let $u = kt$; as $t \to 0, u \to 0$ & as $t \to \infty, u \to \infty$

$$\therefore F(ks) = \int_0^\infty e^{-su} f\left(\frac{u}{k}\right) d\left(\frac{u}{k}\right)$$

$$= \frac{1}{k} \int_0^\infty e^{-su} f\left(\frac{u}{k}\right) du$$

$$= \frac{1}{k} \mathcal{L}\left[f(t/k)\right]$$

$$\therefore \mathcal{L}^{-1}\left[F(ks)\right] = \frac{1}{k} f(t/k)$$

Q: If $\mathcal{L}^{-1}\left[\frac{e^{-1/s}}{s^{1/2}}\right] = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find $\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{1/2}}\right]$ where $a > 0$.

$$\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{1/2}}\right] = \frac{\mathcal{L}^{-1}}{a^{1/2}}\left[\frac{e^{-\frac{1}{\frac{1}{a}s}}}{\left(\frac{1}{a}s\right)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{a}} \cdot \frac{1}{1/a} \cdot \frac{\cos 2\sqrt{\frac{t}{1/a}}}{\sqrt{\pi \cdot \left(\frac{t}{1/a}\right) \cdot t}}$$

$$= \frac{\cos\left(2\sqrt{at}\right)}{\sqrt{\pi t}}$$

E. Inverse Laplace Transform of Derivatives

Theorem 2.6 – If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}[F^{(n)}(s)] = \mathcal{L}^{-1}\left[\frac{d^n}{ds^n}F(s)\right]$$

$$= (-1)^n t^n f(t)$$

Example :- $\mathcal{L}\left[\frac{s}{(s^2+1)^2}\right] = ?$

Since $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$ and $\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{-2s}{(s^2+1)^2}$

we have

$$\mathcal{L}^{-1}\left[\frac{-2s}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right] = (-1)^1 t^1 \sin t$$

$$= -t\sin t$$

or $\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}t\sin t$

Proof :- We have already proved theorem 1.12

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s) = (-1)^n F^{(n)}(s)$$

$$\therefore \mathcal{L}^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$$

F. Inverse Laplace Transform of Integrals

Theorem 2.7 – If $\mathcal{L}^{-1}[F(s)] = f(t)$ then

$$\mathcal{L}^{-1}\left[\int_s^\infty F(u)\,du\right] = \frac{f(t)}{t}$$

Example : Find $\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s}\right)\right]$

$$\mathcal{L}^{-1}\left[\ln\left(\frac{1+s}{s}\right)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\ln(1+s) - \ln(s)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[-\int_s^\infty \frac{1}{1+u}\,du + \int_s^\infty \frac{1}{u}\,du\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\underbrace{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t} = f(t)$$

$$\therefore \mathcal{L}^{-1}\left[\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du\right] = \frac{1-e^{-t}}{t}$$

Q:- Find $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

We have. $\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) = \frac{-2s}{(s^2+a^2)^2}$

Thus, $\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)$

$\therefore \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -\frac{1}{2}(-1)^1 t^1 \frac{\sin(at)}{a}$

$= \frac{t\sin(at)}{2a}$

since $\mathcal{L}\left[\frac{a}{s^2+a^2}\right] = \sin(at)$

Another method : differentiating w.r.t. parameter 'a'

$$\frac{d}{da}\left(\frac{s}{s^2+a^2}\right) = \frac{-2as}{(s^2+a^2)^2}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)\right] = -2a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \frac{d}{da}\left\{\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]\right\} = -2a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -\frac{1}{2a}\frac{d}{da}\left\{\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]\right\}$$

$$= -\frac{1}{2a}\cdot\frac{d}{da}\{\cos(at)\}$$

$$= -\frac{1}{2a}\cdot(-t)\sin(at)$$

$$= \frac{t\sin(at)}{2a}$$

Q: Find $\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]$

$$F(s) = \ln\left(1+\frac{1}{s^2}\right) = \ln\left(\frac{1+s^2}{s^2}\right) = \ln(1+s^2) - \ln(s^2)$$

$$\frac{d}{ds}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\} = \frac{d}{ds}\{\ln(1+s^2)\} - \frac{d}{ds}(\ln s^2)$$

$$= \frac{2s}{1+s^2} - \frac{2s}{s^2}$$

$$= 2\left\{\frac{s}{s^2+1} - \frac{1}{s}\right\}$$

We know $\mathcal{L}^{-1}[F'(s)] = (-1)t f(t)$.

$$\therefore \mathcal{L}^{-1}\left[\frac{d}{ds}\left(\ln\left(1+\frac{1}{s^2}\right)\right)\right] = -t f(t)$$

$$\Rightarrow 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s}\right] = -t f(t)$$

$$\Rightarrow \mathcal{L}\{\cos(t) - 1\} = -t\, f(t)$$

$$\Rightarrow f(t) = \frac{2\{1 - \cos(t)\}}{t}$$

$$\Rightarrow \mathcal{L}^{-1}\left[F(s)\right] = \frac{2\{1 - \cos(t)\}}{t}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\ln\left(1 + \frac{1}{s^2}\right)\right] = \frac{2\{1 - \cos(t)\}}{t}$$

G. Multiplication by s

Theorem 2.8 – If $\mathcal{L}^{-1}[F(s)] = f(t)$ and $f(0) = 0$

$$\mathcal{L}^{-1}[sF(s)] = f'(t)$$

* Multiplication by s has an effect of differentiating

If $f(0) \neq 0$, then

$$\mathcal{L}^{-1}[sF(s) - f(0)] = f'(t)$$

$$\Rightarrow \mathcal{L}^{-1}[sF(s)] = f'(t) + f(0)\underbrace{\delta(t)}_{\text{Dirac Delta function}}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \mathcal{L}^{-1}\left[s \cdot \frac{1}{s^2+1}\right] = \frac{d}{dt}(\sin t) = \cos t$$

H. Division by s

Theorem 2.9 – If $\mathcal{L}^{-1}[F(s)] = f(t)$ then

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du$$

* Division by s has the effect of integrating $f(t)$ from 0 to t.

Proof :- Let $g(t) = \int_0^t f(u)\,du$, $g(0)=0$

$$\mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - \underbrace{g(0)}_{\to 0} = F(s)$$

$$\therefore \mathcal{L}[g(t)] = \frac{F(s)}{s} \quad \therefore \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = g(t) = \int_0^t F(u)\,du$$

In general, $\mathcal{L}^{-1}\left[\frac{F(s)}{s^n}\right] = \int_0^t\int_0^t \cdots \int_0^t f(t)\,dt^n$

Q: Evaluate $\mathcal{L}^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$ using theorem 2.9.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin(t).$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+1)}\right] = \int_0^t du \int_0^u dv \int_0^v dw \,\sin(w)$$

$$= \int_0^t du \int_0^u dv \{-\cos(v) + \cos(0)\}$$

$$= \int_0^t du \int_0^u dv \{1 - \cos(v)\}$$

$$= \int_0^t du \,(u - 0 - \sin(u) - 0)$$

$$= \frac{t^2}{2} - 0 + \cos(t) - \cos(0)$$

$$= \frac{t^2}{2} + \cos(t) - 1$$

H. The Convolution Property.

Theorem 2.10 – If $\mathcal{L}^{-1}[F(s)] = f(t)$,

and $\mathcal{L}^{-1}[G(s)] = g(t)$,

then
$$\mathcal{L}^{-1}[F(s).G(s)] = \int_0^t f(u)g(t-u)\,du = f*g$$

* $f*g$ is called the convolution of f and g.
OR
convolution integral

Proof :- We can prove the convolution theorem if we prove that
$$\mathcal{L}\left[\int_0^t f(u)g(t-u)\,du\right] = F(s)G(s)$$

where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$

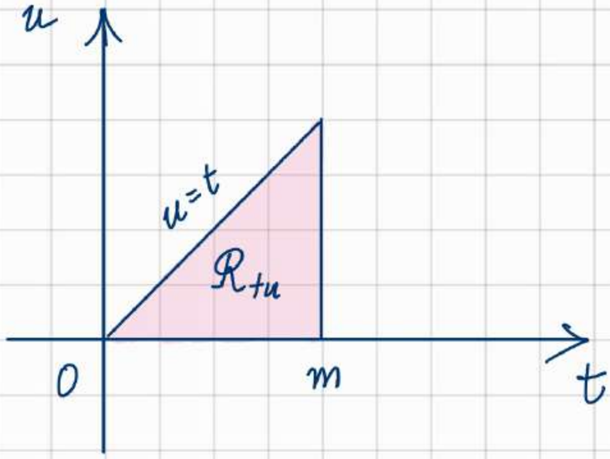
We may write the LHS using the definition of Laplace transform

$$\int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} f(u)g(t-u)\,du\right\}dt$$

$$= \int_{t=0}^{\infty} dt \int_{u=0}^{t} e^{-st} f(u)g(t-u)\,du$$

$$= \lim_{m \to \infty} \int_{t=0}^{m} dt \int_{u=0}^{t} e^{-st} f(u) g(t-u) du$$

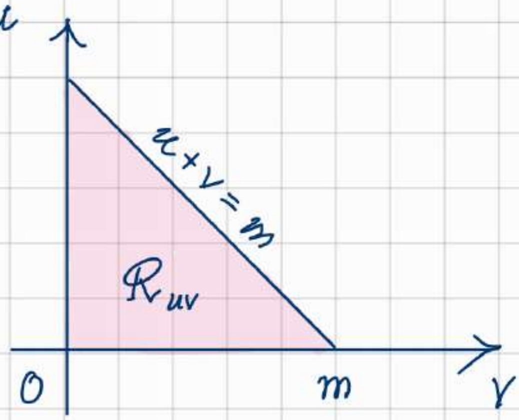
Let us try to visualize the region in the t-u plane over which this integration is performed.



We would like to make the following transformation.

Let $t - u = v$ or $t = u + v$

The region over which the integration is performed changes i.e. $R_{tu} \rightarrow R_{uv}$



The integral may be changed accordingly by using the Jacobian.

$$\lim_{m\to\infty} \int_{R_{tu}} dt \int du \; e^{-st} f(t) g(t-u)$$

$$= \lim_{m\to\infty} \int_{R_{uv}} dv \int e^{-s(u+v)} f(u)\, g(v) \cdot \left| \frac{\partial(u,t)}{\partial(u,v)} \right| du$$

where the Jacobian of the transformation is:

$$J = \left| \frac{\partial(u,t)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

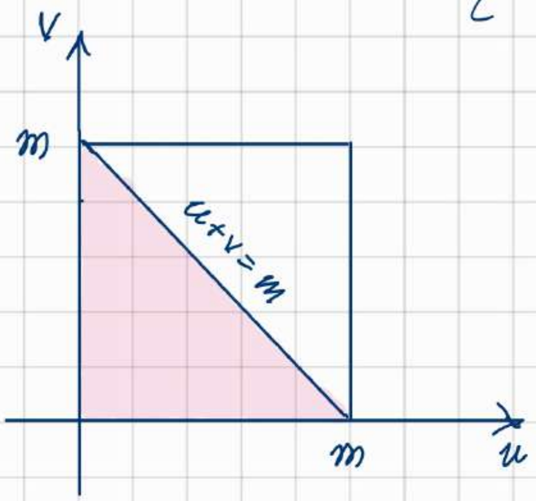
$$= 1$$

Thus, the integral is

$$\lim_{m\to 0} \int_{v=0}^{m} dv \int_{u=0}^{m-v} e^{-s(u+v)} f(u) g(u)\, du$$

Let us define a new function :-

$$K(u,v) = \begin{cases} e^{-s(u+v)} f(u) g(v) & \text{if } u+v \leq m \\ 0 & \text{if } u+v > m \end{cases}$$



This function is defined over the square $(u \in [0,m], v \in [0,m])$

but is 0 over the unshaded portion of the square.

In terms of the new function we can write the integral as.

$$\lim_{m \to \infty} \int_{v=0}^{m} dv \int_{0}^{m} K(u,v)\, du$$

$$= \int_{v=0}^{\infty} dv \int_{u=0}^{\infty} du \; e^{-s(u+v)} f(u) g(v)$$

$$= \int_{v=0}^{\infty} e^{-su} f(u)\, du \cdot \int_{u=0}^{\infty} e^{-sv} g(v)\, dv$$

$$= F(s)\, G(s)$$

which establishes the theorem

Q:- If the convolution of f and g is defined as

$$f*g = \int_0^t f(u)\,g(t-u)\,du$$

then prove that convolution of f and g is commutative.

Let, $t-u = v$ or $u = t-v$ | $du = -dv$

as $u \to 0$, $v \to t$ | as $u \to t$, $v \to 0$

$$\therefore f*g = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_t^0 f(t-v)\,g(v)\,(-dv)$$

$$= \int_0^t g(v)\,f(t-v)\,dv$$

$$= g*f.$$

Q:- Evaluate each of the following by use of the convolution theorem :-

a) $\mathcal{L}^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

$$\frac{s}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)} \cdot \frac{s}{(s^2+a^2)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin(at)}{a} \; ; \; \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$$

By convolution theorem,

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t \frac{\sin(au)}{a} \cos(a(t-u))\, du$$

$$= \int_0^t \frac{\sin(au)}{a} \left(\cos(at)\cos(au) - \sin(at)\sin(au)\right) du$$

$$= \frac{1}{a}\cos(at)\int_0^t \cos(au)\sin(au)\, du$$

$$- \frac{1}{a}\sin(at)\int_0^t \sin^2(au)\, du$$

$$= \frac{1}{2a}\cos(at)\int_0^t \sin(2au)\, du - \frac{1}{2a}\sin(at)\int_0^t (\cos(2au)-1)\, du$$

$$= \frac{1}{2a}\cos(at)\left[-\frac{\cos(2au)}{2a}\right]_0^t - \frac{1}{2a}\sin(at)\left[\frac{\sin(2au)}{2a} - u\right]_0^t$$

$$= \frac{1}{4a^2}\cos(at)\left\{1-\cos(2at)\right\} - \frac{1}{2a}\sin(at)\left\{\frac{\sin(2at)}{2a} - t\right\}$$

$$= \frac{1}{4a^2}\cos(at)\,2\sin^2(at) - \frac{2}{4a^2}\sin^2(at)\cos(at)$$

$$+ \frac{t}{2a}\sin(at)$$

$$= \frac{t\sin(at)}{2a}$$

b) $\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$

$$\frac{1}{s^2(s+1)^2} = \frac{1}{(s+1)^2}\cdot\frac{1}{s^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t} \quad \Big| \quad \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

By convolution theorem,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$$

$$= \int_0^t ue^{-u}(t-u)\,du$$

$$= t\int_0^t u e^{-u}\,du - \int_0^t u^2 e^{-u}\,du$$

$$= t\left\{ u\int e^{-u}\,du \Big|_0^t - \int_0^t \left(1\int e^{-u}\,du\right)du \right\}$$

$$- \left\{ u^2 \int e^{-u}\,du \Big|_0^t - 2\int_0^t \left(v\int e^{-v}\,dv\right)dv \right\}$$

$$= t\left\{ -ue^{-u}\Big|_0^t + \int_0^t e^{-u}\,du \right\} + u^2 e^{-u}\Big|_0^t - 2\int_0^t u e^{-u}\,du$$

$$= t\left\{-te^{-t} - e^{-t} + 1\right\} + t^2 e^{-t} - 2\left\{-te^{-t} - e^{-t} + 1\right\}$$

$$= -\cancel{t^2 e^{-t}} - te^{-t} + t + \cancel{t^2 e^{-t}} + 2te^{-t} + 2e^{-t} - 2$$

$$= te^{-t} + 2e^{-t} + t - 2.$$

I. Methods of finding inverse Laplace transforms.

1. Partial Fractions Method.

Any rational function $\frac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials, with the degree of $P(s)$ less than that of $Q(s)$ can be written as the sum of rational functions (called partial fractions). having the form :-

$$\frac{A}{(as+b)^r}, \quad \frac{As+B}{(as^2+bs+c)^r}, \ldots$$

By finding the inverse Laplace transforms of each of the partial fractions we can find,

$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right].$$

2. Series Method

If $F(s)$ has a series expansion in inverse powers of s given by.

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \cdots$$

then under suitable conditions we can perform inverse Laplace transform term by term to obtain -

$$f(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \cdots$$

3. Miscallaneous Methods.

4. The Complex Inversion formula.

This formula supplies a powerful direct method for finding inverse Laplace transforms and makes use of complex variable theory

Q: Find the following inverse Laplace transforms using the method of partial fractions.

a). $\mathcal{L}^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]$

$$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

Multiplying both sides by $s-3$ and taking limit $s \to 3$

$$\lim_{s \to 3} \frac{3s+7}{s+1} = A + \lim_{s \to 3} \frac{B(s-3)}{s+1}$$

$$\Rightarrow A = \lim_{s \to 3} \frac{3s+7}{s+1} = 4$$

Similarly multiplying both sides by $s+1$ and taking limit $s \to -1$

$$\lim_{s \to -1} \frac{3s+7}{s-3} = \lim_{s \to -1} \frac{A(s+1)}{s-3} + B$$

$$\Rightarrow B = \lim_{s \to -1} \frac{3s+7}{s-3} = -1.$$

Thus, $$\frac{3s+7}{s^2-2s-3} = \frac{4}{s-3} - \frac{1}{s+1}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = 4\,\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$= 4e^{3t} - e^{-t}$$

b). $\mathcal{L}^{-1}\left[\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right]$

Let, $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)} = \dfrac{A}{s+1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$$A = \lim_{s\to -1} \frac{2s^2-4}{(s-2)(s-3)} = -\frac{1}{6}$$

$$B = \lim_{s\to 2} \frac{2s^2-4}{(s+1)(s-3)} = -\frac{4}{3}$$

$$C = \lim_{s\to 3} \frac{2s^2-4}{(s+1)(s-2)} = \frac{7}{2}$$

Thus, $\mathcal{L}^{-1}\left[\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right] = -\dfrac{1}{6}\mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right] - \dfrac{4}{3}\mathcal{L}^{-1}\left[\dfrac{1}{s-2}\right] + \dfrac{7}{2}\mathcal{L}^{-1}\left[\dfrac{1}{s-3}\right]$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

c) $\mathcal{L}^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Let, $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3} = \dfrac{A}{s+1} + \dfrac{B}{(s-2)^3} + \dfrac{C}{(s-2)^2} + \dfrac{D}{(s-2)}$

Multiplying by $s+1$ on both sides and taking limit $s \to -1$

$$A = \lim_{s\to -1} \frac{5s^2-15s-11}{(s-2)^3} = -\frac{1}{3}$$

Multiplying by $(s-2)^3$ on both sides and taking limit $s \to 2$

$$B = \lim_{s\to 2} \frac{5s^2-15s-11}{(s+1)} = -7$$

We can not use this method to find C and D.

Let us multiply s on both sides and take limit $s \to 0$

$$\lim_{s\to\infty} \frac{s(5s^2-15s-11)}{(s+1)(s-2)^3} = \lim_{s\to\infty}\left(-\frac{1}{3}\cdot\frac{s}{s+1}\right) + \lim_{s\to\infty} -\frac{7s}{(s-2)^3}$$

$$+ \lim_{s\to\infty} \frac{Cs}{(s-2)^2} + \lim_{s\to\infty} \frac{Ds}{s-2}$$

$$\lim_{s\to\infty} \frac{s \cdot s^3\left(\frac{5}{s} - \frac{15}{s^2} - \frac{11}{s^3}\right)}{s\left(1+\frac{1}{s}\right)\cdot s^3\left(1-\frac{2}{s}\right)^3} = \lim_{s\to\infty} \frac{5/s - 15/s^2 - \frac{11}{s^3}}{\left(1+1/s\right)\left(1-\frac{2}{s}\right)^3} = 0$$

$$\lim_{s\to\infty}\left(-\frac{1}{3}\,\frac{s}{s+1}\right) = \lim_{s\to\infty}\left(-\frac{1}{3}\,\frac{1}{1+1/s}\right) = -\frac{1}{3}$$

$$\lim_{s\to\infty}\left(-\frac{7s}{(s-2)^3}\right) = \lim_{s\to\infty}\left(-\frac{s^3\cdot 7/s^2}{s^3\left(1-\frac{2}{s}\right)^3}\right) = 0$$

$$\lim_{s\to\infty}\left(\frac{Cs}{(s-2)^3}\right) = \lim_{s\to\infty}\left(\frac{s^3\, C/s^2}{s^3\left(1-2/s\right)^3}\right) = 0$$

$$\lim_{s\to\infty}\left(\frac{Ds}{s-2}\right) = \lim_{s\to\infty}\left(D\cdot\frac{1}{1-2/s}\right) = D.$$

This gives, $0 = -\frac{1}{3} + D \Rightarrow D = \frac{1}{3}$

As we know A, B and D, we can find C.

$$\frac{(5s^2-15s-11)}{(s+1)(s-2)^3} = \left(-\frac{1}{3}\cdot\frac{1}{s+1}\right) - \frac{7}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{1}{3}\left(\frac{1}{s-2}\right)$$

Setting $s=0$, $\frac{-11}{-8} = -\frac{1}{3} - \frac{7}{-8} + \frac{C}{4} + \frac{1}{3}\left(\frac{1}{-2}\right)$

$$\Rightarrow -11 = \frac{8}{3} - 7 - 2C + \frac{4}{3}$$

$$\Rightarrow 2C = 4 - 7 + 11$$

$$\Rightarrow C = 4.$$

Thus, $$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{(-1/3)}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{1/3}{(s-2)}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = -\frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 7\mathcal{L}^{-1}\left[\frac{1}{(s-2)^3}\right] + 4\left[\frac{1}{(s-2)^2}\right] + \frac{1}{3}\left[\frac{1}{s-2}\right]$$

$$= -\frac{1}{3}e^{-t} - 7t^2e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}$$

d) $$\mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$$

Let, $$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

Multiplying both sides by $(s-1)$ and taking limit $s \to 1$

$$A = \lim_{s\to 1} \frac{3s+1}{s^2+1} = 2$$

Setting $s=0$,

$$\frac{3.0+1}{(0-1)(0^2+1)} = \frac{2}{0-1} + \frac{B.0+C}{0^2+1}$$

$$\Rightarrow -1 = -2 + C$$

$$\Rightarrow C = 3$$

Setting $s=2$. (something other than 0 & 1)

$$\frac{3.2+1}{(2-1)(4+1)} = \frac{2}{2-1} + \frac{2B+3}{4+1}$$

$$\Rightarrow \frac{7}{5} = 2 + \frac{2B+3}{5}$$

$$\Rightarrow 2B+3 = 7-10$$

$$\Rightarrow 2B = -6 \Rightarrow B = -2.$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+3}{s^2+1}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^t - 2\cos(t) + 3\sin(t)$$

J. Heavyside Expansion Formula.

Let $P(s)$ and $Q(s)$ be polynomials where $P(s)$ has degree less than that of $Q(s)$.

Suppose $Q(s)$ has n distinct zeros. α_k, $(k = 1, 2, 3, \ldots n)$

Then the inverse Laplace transform of $F(s) = \dfrac{P(s)}{Q(s)}$ is given by -

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

Proof :- Since $Q(s)$ is a polynomial with n distinct zeros $\alpha_1, \alpha_2, \alpha_3 \cdots, \alpha_n$, we can write $\dfrac{P(s)}{Q(s)}$ as the sum of partial fractions as -

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \cdots + \frac{A_k}{s-\alpha_k} + \cdots + \frac{A_n}{s-\alpha_n}$$

Multiplying both sides by $(s-\alpha_k)$ and letting $s \to \alpha_k$ we can find the coefficient A_k as.

$$A_k = \lim_{s \to \alpha_k} \frac{P(s)}{Q(s)} (s-\alpha_k)$$

$$A_k = \lim_{s \to \alpha_k} P(s) \left\{ \frac{s - \alpha_k}{Q(s)} \right\}$$

$$= \lim_{s \to \alpha_k} P(s) \cdot \lim_{s \to \alpha_k} \left\{ \frac{s - \alpha_k}{Q(s)} \right\}$$

To find this limit we shall use L'Hopital's rule,

$$A_k = \lim_{s \to \alpha_k} P(s) \cdot \lim_{s \to \alpha_k} \frac{\frac{d}{ds}(s - \alpha_k)}{\frac{d}{ds} Q(s)}$$

$$= \lim_{s \to \alpha_k} P(s) \cdot \lim_{s \to \alpha_k} \frac{1}{Q'(s)}$$

$$= P(\alpha_k) \cdot \frac{1}{Q'(\alpha_k)}$$

Thus, $F(s) = \frac{P(s)}{Q(s)}$ may be written as :

$$F(s) = \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot \frac{1}{(s - \alpha_1)} + \cdots + \frac{P(\alpha_k)}{Q'(\alpha_k)} \cdot \frac{1}{(s - \alpha_k)} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)} \frac{1}{(s - \alpha_n)}$$

Thus, the inverse Laplace transform may be obtained as.

$$\mathcal{L}^{-1}\left[F(s)\right] = \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot e^{\alpha_1 t} + \cdots + \frac{P(\alpha_k)}{Q'(\alpha_k)} \cdot e^{\alpha_k t} + \cdots$$

$$\cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

Q: Find the following inverse Laplace transform using Heavyside's expansion formula -

a) $\mathcal{L}^{-1}\left[\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right]$

Here $P(s) = 2s^2 - 4$

$$Q(s) = (s+1)(s-2)(s-3)$$

$$= s^3 - 4s^2 + s + 6$$

$$Q'(s) = 3s^2 - 8s + 1$$

The zeros of $Q(s)$ are at $-1, 2, 3$.

Thus, $\mathcal{L}^{-1}\left[\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right]$

$$= \frac{P(-1)}{Q'(-1)} e^{-t} + \frac{P(2)}{Q'(2)} e^{2t} + \frac{P(3)}{Q'(3)} e^{3t}$$

$$= \frac{-2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

b) $\mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = \mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s+i)(s-i)}\right]$

$$P(s) = 3s+1$$

$$Q(s) = (s-1)(s^2+1) = s^3 - s^2 + s - 1$$

$$Q'(s) = 3s^2 - 2s + 1$$

The zeros of $Q(s)$ are at 1, $+i$ and $-i$

Thus. $\mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = \frac{P(1)}{Q'(1)}e^{t} + \frac{P(+i)}{Q'(+i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it}$

$$= \frac{4}{2}e^{t} + \frac{3i+1}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

$$\frac{(3i+1)(2-2i)}{-(2+2i)(2-2i)}$$

$$\frac{6i+6+2-2i}{-8}$$

$$\frac{4i+8}{-8} = -1-\frac{1}{2}i$$

$$= 2e^{t} + \left(-1-\tfrac{1}{2}i\right)(\cos t + i\sin t)$$
$$+ \left(-1+\tfrac{1}{2}i\right)(\cos t - i\sin t)$$

$$= 2e^{t} - \cos(t) + \tfrac{1}{2}\sin t - \cos t + \tfrac{1}{2}\sin t$$

$$= 2e^{t} - 2\cos t + \sin(t).$$