The Laplace transformation is useful in solving linear ordinary differential equations commonly arising in Physics and engineering.

By transforming a differential equation for an unknown function $f(t)$ into an algebraic equation for its transform $F(t)$, we can find solutions more efficiently

Laplace Transform is particularly useful for the following types of differential equations.

- Non-homogenous ordinary differential equations with constant coefficients.
- Non-homogenous ordinary differential equations with variable coefficints.
- A system of simultaneous ordinary differential equations.
- A few partial differential equations: boundary value problems.

A. Ordinary Differential Equations with constant coefficients.

Q: Solve $y''(t) + y(t) = t$; $y(0) = 1$; $y'(0) = -2$

Taking Laplace transformation on both sides

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$$\Rightarrow s^2 Y(s) - s + 2 + Y(s) = \frac{1}{s^2}$$

$$\Rightarrow Y(s)\left(1+s^2\right) = s - 2 + \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)} \qquad \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$= \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

Hence, $y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = t + \cos(t) - 3\sin(t)$

Q :- Solve $y'' - 3y' + 2y = 4e^{2t}$, $y(0) = -3$, $y'(0) = 5$

$$\mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] = 4\mathcal{L}[e^{2t}]$$

$$s^2 Y(s) - s y(0) - y'(0)$$
$$-3\{sY(s) - y(0)\} + 2Y(s) = 4\frac{1}{s-2}$$

$$s^2 Y + 3s - 5 - 3(sY + 3) + 2Y = \frac{4}{s-2}$$

$$\Rightarrow (s^2 - 3s + 2)Y + 3s - 14 = \frac{4}{s-2}$$

$$Y = \frac{4}{(s-2)(s^2 - 3s + 2)} + \frac{14 - 3s}{(s^2 - 3s + 2)}$$

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$$= \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

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$$= \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right]$$

$$= -7e^{t} + 4e^{2t} + 4.te^{2t}$$

* verify !!!

B. Ordinary differential functions with variable coefficients.

Q: Solve $ty'' + y' + 4ty = 0$, $y(0) = 3$, $y'(0) = 0$

$$\mathcal{L}[ty'' + y' + 4ty] = 0$$

$$-\frac{d}{ds}\{s^2 Y - s y(0) - y'(0)\} + \{sY - y(0)\}$$

$$-4\frac{d}{ds}Y = 0$$

$$-\frac{d}{ds}\{s^2 Y - 3s\} + \{sY - 3\} - 4\frac{dY}{ds} = 0$$

$$\Rightarrow -s^2\frac{dY}{ds} - 2sY + 3 + sY - 3 - 4\frac{dY}{ds} = 0$$

$$\Rightarrow (s^2+4)\frac{dY}{ds} + sY = 0$$

$$\Rightarrow \frac{dY}{ds} = -\frac{sY}{(s^2+4)}$$

$$\Rightarrow \frac{dY}{Y} = -\frac{s\,ds}{s^2+4}$$

$$\Rightarrow \ln Y = -\frac{1}{2}\ln(s^2+4) + \ln c$$

$$\Rightarrow \ln Y + \frac{1}{2}\ln(s^2+4) = \ln c$$

$$\Rightarrow \quad \ln\left(Y \cdot \sqrt{s^2+4}\right) = \ln C$$

$$\Rightarrow \quad Y = \frac{c}{\sqrt{s^2+4}}$$

$$\mathcal{L}^{-1}[Y] = c\,\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s^2+4}}\right] = ?$$

C. Damped Simple Harmonic Oscillator

A. Harmonic Oscillator

$$y'' + \omega^2 y = 0$$

$$s^2Y - sy(0) - y'(0) + \omega^2 Y = 0 .$$

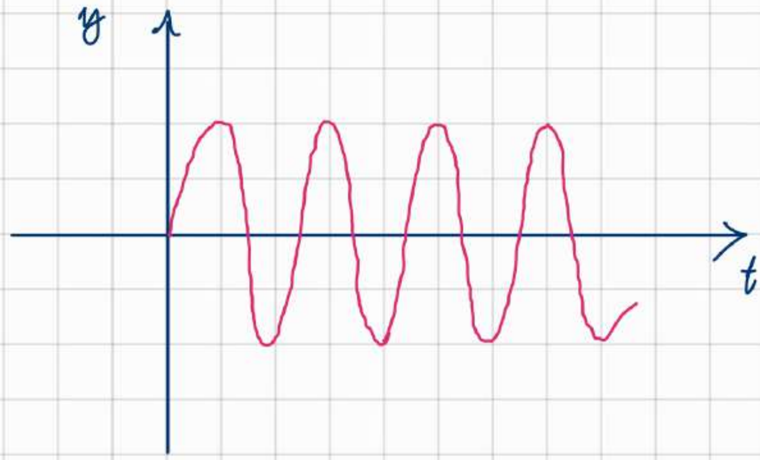
$$s^2Y - As - 0 + \omega^2 Y = 0$$

$$\Rightarrow (s^2 + \omega^2) Y = As$$

$$\Rightarrow Y = \frac{As}{s^2 + \omega^2}$$

$$\mathcal{L}^{-1}[Y] = A\,\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right]$$

$$y(t) = A\cos(\omega t)$$



B. Damped Simple Harmonic Oscillator.

$$y'' - 2\alpha y' + \omega^2 y = 0$$

where $-2\alpha y'$ is the damping term and the system is subjected to the initial conditions $y(0) = y_0$, $y'(0) = v_0$.

The Laplace transform of this equation using the above initial conditions is -

$$s^2 Y - y_0 s - v_0 + 2\alpha(sY - y_0) + \omega^2 Y = 0$$

$$\Rightarrow Y = \frac{sy_0 + (v_0 + 2\alpha y_0)}{s^2 + 2\alpha s + \omega^2}$$

$$= \frac{(s+\alpha)y_0}{(s+\alpha)^2 + \omega^2 - \alpha^2} + \frac{v_0 + \alpha y_0}{(s+\alpha)^2 + \omega^2 - \alpha^2}$$

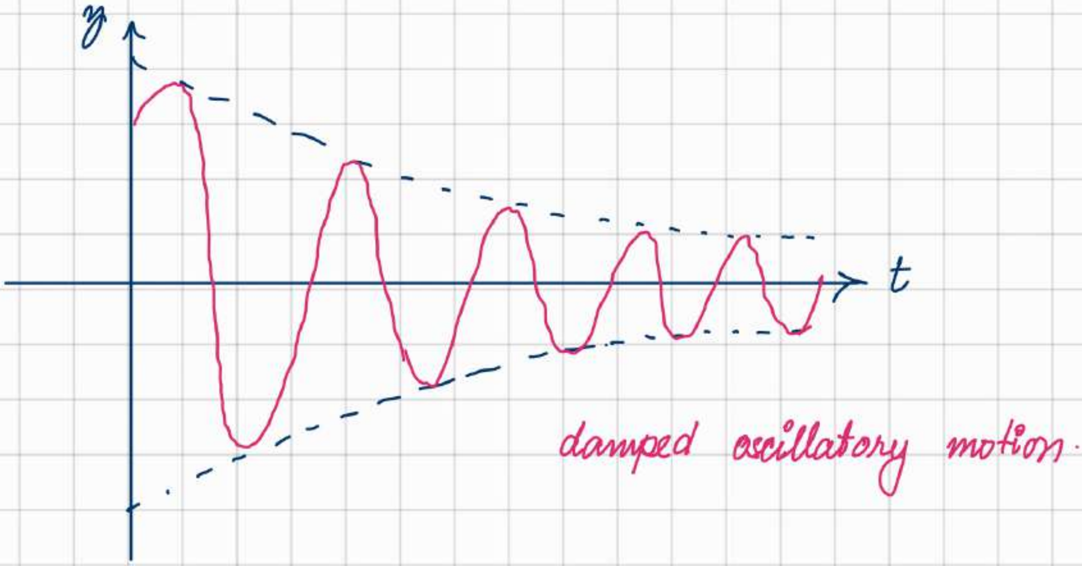
Case I :- $\omega^2 - \alpha^2 > 0$

In this case,

$$y(t) = \mathcal{L}^{-1}[Y] = y_0 e^{-\alpha t} \cos\left(\sqrt{\omega^2 - \alpha^2}\, t\right)$$

$$+ \frac{v_0 + \alpha y_0}{\sqrt{\omega^2 - \alpha^2}} e^{-\alpha t} \sin\left(\sqrt{\omega^2 - \alpha^2}\, t\right)$$

This motion is called damped oscillatory.



The particle oscillates about O, the magnitude of each oscillation becoming smaller with each swing.

The period of the oscillations is given by $\dfrac{2\pi}{\sqrt{\omega^2-\alpha^2}}$ and thus the frequency is $\dfrac{\sqrt{\omega^2-\alpha^2}}{2\pi}$

The quantity $\omega/2\pi$ (corresponding to $\alpha=0$ i.e. no damping) is called the natural frequency.

Case II :- $\omega^2-\alpha^2=0$

In this case,

$$y(t) = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{y_0}{s+\alpha} + \frac{v_0+\alpha y_0}{(s+\alpha)^2}\right]$$

$$= y_0 e^{-\alpha t} + (v_0+\alpha y_0)\, t e^{-\alpha t}$$

Here the particle does not oscillate indefinitely about O. Instead, it approaches O gradually but never reaches it.

The motion is called critically damped motion since any decrease in the damping constant 2α would produce oscillations.

critically damped motion

Case III : $\omega^2 - \alpha^2 < 0$

In this case,

$$y(t) = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{(s+\alpha)y_0}{(s+\alpha)^2 - (\alpha^2 - \omega^2)} + \frac{v_0 + \alpha y_0}{(s+\alpha)^2 - (\alpha^2 - \omega^2)}\right]$$

$$= y_0 \cosh\left(\sqrt{\alpha^2 - \omega^2}\, t\right) + \frac{v_0 + \alpha y_0}{\sqrt{\alpha^2 - \omega^2}} \cdot \sinh\left(\sqrt{\alpha^2 - \omega^2}\, t\right)$$

This motion is called overdamped motion and is non-oscillatory. The graph is similar to that of a critically damped motion.