Fourier Transform

In our previous semester we encountered the Fourier series representation of a periodic function in a fixed interval as a superposition of sinusoidal functions.

We also want to obtain such a representation even for functions defined over an infinite interval and with no particular periodicity. Such a transformation is called a Fourier transform.

The Fourier transform of a function $f(t)$ is defined as

$$\boxed{F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \, dt}$$

As such the inverse Fourier transform is defined as -

$$\boxed{f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} \, d\omega}$$

Example 1 : Find the Fourier transform of the exponential decay function $f(t) = 0$ for $t < 0$ and $f(t) = Ae^{-\lambda t}$ for $t \geq 0$ $(\lambda > 0)$.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} 0 \cdot e^{-i\omega t} dt + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t} e^{-i\omega t} dt$$

$$= 0 + \frac{A}{\sqrt{2\pi}} \left[\frac{-e^{-(\lambda + i\omega)t}}{\lambda + i\omega} \right]_0^{\infty}$$

$$= \frac{A}{\sqrt{2\pi}} \left(\frac{1}{\lambda + i\omega} \right)$$

How do we go from Fourier Series to Fourier Transform?

The complex form of Fourier Series. —

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}$$

where C_n are coefficients given by.

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega_n t} dt$$

* $\omega_n = \frac{2\pi n}{T}$ / where $\Delta\omega = \frac{2\pi}{T}$

In terms of $\Delta\omega$, C_n can be written as -

$$C_n = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-i\omega_n t}\,dt$$

If $T \to \infty$, then $\Delta\omega \to 0$; $d\omega$

So, the above equation may be written as -

$$C_n = \frac{d\omega}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\omega_n t}\,dt$$

Suppose we replace C_n by $g(\omega)$,

$$g(\omega) = \frac{d\omega}{2\pi}\int_{-\infty}^{+\infty} f(t)\,e^{-i\omega_n t}\,dt$$

as under $T \to \infty$, C_n becomes a function of ω only because C_n in this case becomes continuous with respect to ω.

This means,

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t} \longrightarrow \int_{-\infty}^{+\infty} g(\omega)\,e^{i\omega t}\,d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\left(e^{i\omega t}\int_{-\infty}^{+\infty} f(t')\,e^{-i\omega t'}dt'\right)$$

This is known as Fourier Inversion theorem.

From this theorem we can define fourier transform as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

and inverse Fourier transform as -

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} dt$$

* It should be noted that for function $f(t)$ to be Fourier transformable it must satisfy.

$$\int_{-\infty}^{+\infty} |f(t)| \, dt = \text{finite}$$

Let us try to calculate the integration,

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{+\infty} \underbrace{f^*(t) f(t)}_{} \, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} F^*(\omega')e^{-i\omega' t}\,\underline{\underline{d\omega'}}\right\}\left\{\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,\underline{\underline{d\omega}}\right\} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(\omega-\omega')t}\,dt \int_{-\infty}^{+\infty} F^*(\omega')\,d\omega' \int_{-\infty}^{+\infty} F(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\,\delta(\omega-\omega')\int_{-\infty}^{+\infty} F^*(\omega')\,d\omega' \int_{-\infty}^{+\infty} F(\omega)\,d\omega$$

$$\hookrightarrow \delta(\omega-\omega') = 1 \text{ if } \omega=\omega'$$
$$= 0 \text{ if } \omega \neq \omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(\omega')\,d\omega' \int_{-\infty}^{+\infty} F(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2\,d\omega$$

$$\boxed{\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2\,d\omega}$$

This result is known as Parseval's Theorem.

important

Parseval Theorem :-

This theorem states that the integration of the square modulus a function over the whole range is equal to the integration of the squre modulus of the Fourier transformed function over the whole range.

Dirac Delta Function and Fourier Transform :-

Dirac Delta Function is defined by the properties -

$$\delta(t) = 0 \quad \text{if } t \neq 0$$
$$= \infty \quad \text{if } t = 0$$

and
$$\int_a^b \delta(t)\,dt = 1$$

and.
$$\int_{b_1}^{b_2} f(t)\,\delta(t-a)\,dt = f(a) \longrightarrow (i)$$

From the Fourier Inversion theorem,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t}\,d\omega \left\{ \int_{-\infty}^{+\infty} f(t')\,e^{-i\omega t'}\,dt' \right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f'(t) \left[\int_{-\infty}^{+\infty} e^{i\omega(t-t')}\,d\omega \right] dt \rightarrow (2)$$

Comparing ① and ②, we may write.

$$\delta(t-a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t-a)}\,d\omega$$

If $a=0$.

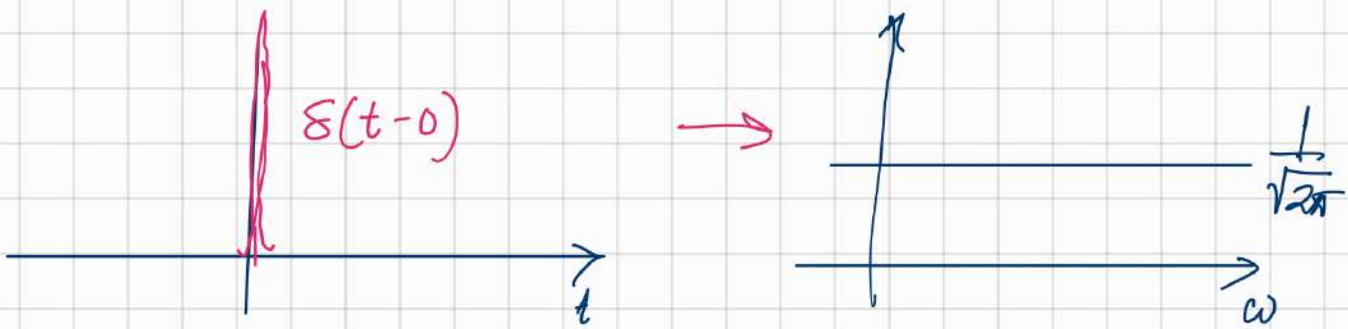
$$\boxed{\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\,d\omega}$$ → definition

Fourier transform of Dirac Delta Function ?

$$F\big(\delta(t)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \delta(t)\, e^{-i\omega t}\,dt$$

fourier transform

$$= \frac{1}{\sqrt{2\pi}}$$ "constant function"



Fourier Transform of Odd and Even Functions

Even function: $f(-x) = f(x)$

eg: x^2, $|x|$, $\cos(x)$

Odd function: $f(-x) = -f(x)$

eg: x^3, $\sin(x)$

For Even function,

Fourier transform of an even function –

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f(t)e^{-i\omega t}\,dt + \int_{0}^{\infty} f(t)e^{-i\omega t}\,dt\right]$$

$f(t)$ even
as $t \to -t$
$dt \to -dt$
$f(t) \to +f(t)$
$e^{-i\omega t} \to e^{i\omega t}$
$-\int_{+\infty}^{0} f(t)e^{i\omega t}\,dt = \int_{0}^{\infty} f(t)e^{i\omega t}\,dt$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} f(t)e^{i\omega t}\,dt + \int_{0}^{\infty} f(t)e^{-i\omega t}\,dt\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\int_{0}^{\infty} f(t)\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right)dt\right.$$

$$\boxed{F(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos(\omega t)\,dt}$$

F.T. of even function

Similarly for odd function.

Fourier transform of an odd function $f(t)$;

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \underbrace{f(t) e^{-i\omega t}}_{t \to -t} dt + \int_{0}^{\infty} f(t) e^{-i\omega t} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{+\infty}^{0} -f(t) e^{-i\omega(-t)} (-dt) + \int_{0}^{\infty} f(t) e^{-i\omega t} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{0}^{\infty} f(t) e^{i\omega t} dt + \int_{0}^{\infty} f(t) e^{-i\omega t} dt \right]$$

$$= \frac{2i}{\sqrt{2\pi}} \left[-\int_{0}^{\infty} f(t) \cdot \left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) dt \right.$$

$$= -i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin(\omega t) \, dt$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \{ \cos(\omega t) - i \sin(\omega t) \} dt$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \overbrace{\int_{-\infty}^{+\infty} \underbrace{f(t) \cos(\omega t)}_{\text{odd} \times \text{even} = \text{odd}} dt}^{0} - i \int_{-\infty}^{+\infty} f(t) \cdot \sin(\omega t) dt \right\}$$

$$= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \sin(\omega t)\, dt$$

F.T. of even function. $f(t)$

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos(\omega t)\, dt$$

Note: Fourier transform of an even real function $f(t)$ is real.

F.T. of odd function $f(t)$

$$F(\omega) = -i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin(\omega t)\, dt$$

Note: Fourier transform of an odd real function, $f(t)$ is purely imaginary.

Fourier Transformation of Derivatives and Integrals

F.T. of $f(t)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

We want to find the fourier transform of derivative of $f(t)$ i.e. $\frac{df}{dt}$

Let, us denote the F.T. of nth derivative as

$$F_n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^n f}{dt^n} \cdot e^{-i\omega t} dt$$

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df}{dt} e^{-i\omega t} dt$$

$$\left(\int u \, dv \quad u = e^{-i\omega t} \quad dv = \frac{df}{dt} \cdot dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{-i\omega t} \cdot f(t) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t) \frac{d}{dt}\left(e^{-i\omega t}\right) dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{e^{-i\omega t} f(t) \Big|_{-\infty}^{+\infty}}^{\,0} + i\omega \int_{-\infty}^{+\infty} e^{-i\omega t} f(t) \, dt \right]$$

$f(t) \to 0$ as $t \to \pm\infty$

$$= \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt$$

$$= i\omega\, F(\omega)$$

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df}{dt} e^{-i\omega t} dt = i\omega\, F(\omega)$$

$$F_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^2 f}{dt^2} e^{-i\omega t} dt = (i\omega)^2 F(\omega)$$

$$- - - - -$$

$$\boxed{F_n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^n f}{dt^n} e^{-i\omega t} dt = (i\omega)^m F(\omega)}$$

F.T. is defined as -

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

We want to find the F.T. of $g(t) = \int f(t) dt$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left\{ \int f(t) dt + C \right\} e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int f(t) dt \right) e^{-i\omega t} dt$$

$$+ \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} dt$$

$\delta(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} dt$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{\int f(t) dt}\, \frac{e^{-i\omega t}}{-i\omega} \Bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \left(f(t) \cdot \frac{e^{-i\omega t}}{-i\omega} \right) dt \right\}$$

$\int f(t) dt \to 0$ at $\pm\infty$ as we are taking its F.T

$$+ C\,\delta(\omega)$$

$$= \frac{1}{i\omega} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt \;+\; C\,\delta(\omega)$$

$$\mathcal{F}\left[\int f(t)\,dt\right] = \frac{1}{i\omega} F(\omega) + C\,\delta(\omega)$$

Properties of Fourier Transform.

(i) Scaling.

$$F\left[f(at)\right] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

where $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$

(ii) Translation / Shifting.

$$F\left[f(t+a)\right] = e^{i\omega a} F(\omega)$$

where $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$

(iii) Multiplication by an exponential term.

$$F\left[e^{\alpha t} f(t)\right] = F\left[\omega + i\alpha\right]$$

$$\frac{1}{\sqrt{2\pi}} \int e^{\alpha t} f(t) e^{-i\omega t} dt.$$

$$\int f(t)\, e^{-i\left(\omega - \frac{\alpha}{i}\right)t} dt$$

$$\int f(t)\, e^{-i(\omega + i\alpha)t} dt.$$

$$F\left(\omega + i\alpha\right).$$

---

* Convolution theorem.

The convolution of two functions $f(t)$ and $g(t)$ where $-\infty < t < \infty$ is defined as

$$f * g = \int_{-\infty}^{+\infty} f(u)\, g(t-u)\, du = h(t)$$

The convolution theorem for Fourier Transform states that -

the fourier transform of convolution of two functions $f(t)$ and $g(t)$ is the product of the fourier transforms of $f(t)$ and $g(t)$

if $h(t) = f(t) * g(t)$ is the convolution of $f(t)$ & $g(t)$,

defined as - $h(t) = \int_{-\infty}^{+\infty} f(t)\, g(t-u)\, du$,

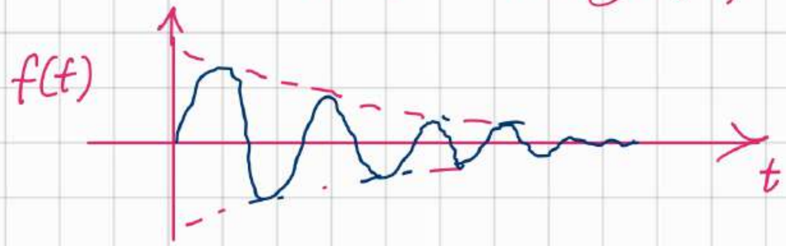
then according to the convolution theorem,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(t)\, e^{-i\omega t} dt = \left\{ \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt \right\} \cdot \left\{ \int_{-\infty}^{+\infty} g(t)\, e^{-i\omega t} dt \right\}$$

$$\mathcal{F}[h(t)] = \mathcal{F}[f(t)] \cdot \mathcal{F}[g(t)]$$

Example : A damped simple harmonic motion may be represented by -

$$f(t) = \begin{cases} 0 & , t < 0 \\ e^{-t/T} \sin(\omega_0 t) , & t \geq 0 \end{cases}$$



Analyse this function in the Fourier space.

The Fourier transform of $f(t)$ is -

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{+\infty} 0 . e^{-i\omega t} dt + \int_0^{\infty} e^{-t/T} \sin(\omega_0 t) e^{-i\omega t} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} e^{-t/T} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right) e^{-i\omega t} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2i} \left\{ \int_0^{\infty} e^{-i\left(-\frac{i}{T} + \omega - \omega_0\right)t} dt - \int_0^{\infty} e^{-i\left(-\frac{i}{T} + \omega + \omega_0\right)t} dt \right\}$$

$$= \frac{1}{2i\sqrt{2\pi}} \left\{ \frac{e^{-i\left(-\frac{i}{T} + \omega - \omega_0\right)}}{-i\left(\frac{-i}{T} + \omega - \omega_0\right)} \Bigg|_0^{\infty} - \right.$$

$$\left.\frac{e^{-i\left(\frac{-i}{T}+\omega+\omega_0\right)}}{-i\left(\frac{-i}{T}+\omega+\omega_0\right)}\right|_0^{\infty}\Bigg\}$$

$$= \frac{1}{2i\sqrt{2\pi}}\left\{0 - \frac{1}{-i\left(\frac{-i}{T}+\omega-\omega_0\right)} - 0 + \frac{1}{-i\left(\frac{-i}{T}+\omega+\omega_0\right)}\right\}$$

$$= \frac{1}{2i\sqrt{2\pi}}\left\{\frac{1}{i\left(\frac{-i}{T}+\omega-\omega_0\right)} - \frac{1}{i\left(\frac{-i}{T}+\omega+\omega_0\right)}\right\}$$

$$= \frac{1}{2i^2\sqrt{2\pi}}\left\{\frac{1}{\left(\frac{-i}{T}+\omega-\omega_0\right)} - \frac{1}{\left(-\frac{i}{T}+\omega+\omega_0\right)}\right\}$$

$$= \frac{1}{2\sqrt{2\pi}}\left\{\frac{1}{\left(\omega+\omega_0-i/T\right)} - \frac{1}{\left(\omega-\omega_0-i/T\right)}\right\}$$

Example : Solve the wave equation

$$\frac{\partial^2}{\partial x^2}y(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}y(x,t)$$

by the method of Fourier transformation. Use the initial conditions

$$y(x,0) = f(x) \text{ and } \frac{dy}{dt}(x,0) = 0$$

Applying Fourier Transform in x to the given eq^n,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\partial^2 y(x,t)}{\partial x^2}\, e^{-i\alpha x}\, dx = \frac{1}{v^2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\partial^2 y(x,t)}{\partial t^2} e^{-i\alpha x} dx$$

$$\Rightarrow (i\alpha)^2\cdot Y(\alpha,t) = \frac{1}{v^2}\frac{d}{dt^2} Y(\alpha,t)$$

This leads to another differential equation which can be written as -

$$\frac{d^2}{dt^2} Y(\alpha,t) + \alpha^2 v^2\, Y(\alpha,t) = 0 \longrightarrow (1)$$

Since the form of this equation is same as the Simple Harmonic Equation, hence the general solution of this equation may be written as -

$$Y(\alpha,t) = A(\alpha)\sin(\alpha v t) + B(\alpha)\cos(\alpha v t) \longleftrightarrow (2)$$

where $A(\alpha)$ and $B(\alpha)$ are two coefficients which are to be determined.

To use the boundary conditions given in the problem, we need to make Fourier Transform of the conditions -

$y(x,0) = f(x)$ transforms as $Y(\alpha,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx$

$\equiv F(\alpha)$ (say)

and $\frac{dy}{dt}(x,0) = 0$ transforms as $\frac{d}{dt} Y(\alpha,0) = 0$.

Applying these boundary conditions to ② we get,

$$Y(\alpha,t) = F(\alpha)\cos(\alpha v t) \rightarrow ⑤$$

The inverse Fourier Transform of ⑤ will give the solution of original wave eq^n which is –

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} Y(\alpha,t) e^{i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) \cos(\alpha v t) e^{i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) \left(\frac{e^{i\alpha v t} + e^{-i\alpha v t}}{2} \right) e^{i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{i(vt+x)\alpha} d\alpha$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{-i(vt-x)\alpha} d\alpha$$

$$\longrightarrow ⑥$$

Inverse Fourier Transform of the boundary condition

$$Y(\alpha,0) = F(\alpha) \quad \text{is}$$

$$y(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{i\alpha x} d\alpha$$

In analogy to this equation, we may write the following equation -

$$f(vt \pm x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{i(vt \pm x)\alpha} d\alpha \rightarrow (7)$$

Comparing equation (6) and (7) we find that the solution of the given equation is -

$$y(x,t) = \frac{1}{2} f(vt + x) + \frac{1}{2} f(vt - x)$$

This is the general form of the wave equation solution.