Matrix Algebra and Index Notation

Matrix algebra is a branch of mathematics that deals with operations on matrices, such as addition, multiplication, inversion, and determinant. Index notation is a convenient way to write the elements and operations of matrices using subscripts.

- A matrix is a rectangular array of numbers arranged in rows and columns. The size of a matrix is given by the number of rows and columns, denoted by m and n respectively. A matrix with m rows and n columns is called an m-by-n matrix, and written as

$$A_{m \times n} \quad \text{or} \quad [A]_{m \times n}$$

- Each element in a matrix can be referenced by its index location, which is given by two numbers: the row number and the column number. For example, the element in the i-th row and j-th column of matrix A is denoted by

$$a_{ij} \quad \text{or} \quad A_{ij}$$

The indices i and j usually range from 1 to m and 1 to n respectively, unless otherwise specified.

- A vector is a special case of a matrix that has only one column or one row. A vector with n elements is called an n-dimensional vector, and written as

$$V_n$$

A vector can be written as a column matrix or a row matrix, depending on the context. For example, the vector V_3 can be written as

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

. The element in the i-th position of vector v is denoted by v_i or V_i

Matrix addition is an operation that adds two matrices of the same size element-wise. That is, the sum of two matrices A and B is a matrix C such that

$$C_{ij} = A_{ij} + B_{ij}$$

for all i and j. Matrix addition is **commutative** and **associative**, meaning that

$$A + B = B + A$$

and

$$(A + B) + C = A + (B + C)$$

for any matrices A, B, and C of the same size.

Identity matrix exists for the addition of matrices, i.e. if A and B are two matrices of the same dimensionality, such that

$$A + B = A$$

then B is the identity matrix for the addition operation. Here B is the null matrix, $B_{ij} = 0$

A **scalar** is a single number that can be multiplied with a matrix or a vector. **Scalar multiplication** is an operation that multiplies each element of a matrix or a vector by a scalar. For example, if α is a scalar and A is a matrix, then αA is a matrix such that

$$(\alpha A)_{ij} = \alpha A_{ij}$$

for all i and j. Scalar multiplication is commutative and associative, meaning that

$$\alpha(\beta A) = (\alpha\beta)A$$

and

$$\alpha(\beta A) = (\alpha A)\beta$$

for any scalars α and β and any matrix A. Scalar multiplication is also distributive over matrix addition, meaning that

$$\alpha(A + B) = \alpha A + \alpha B$$

for any scalar α and any matrices A and B of the same size.

Matrix multiplication is an operation that combines two matrices A and B to produce a matrix C such that

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj}$$

for all i and j, where n is the number of columns of A and the number of rows of B.
Matrix multiplication is not commutative, meaning that

$$AB \neq BA$$

in general, but it is associative, meaning that

$$(AB)C = A(BC)$$

for any matrices A, B, and C that can be multiplied. Matrix multiplication is also distributive over matrix addition, meaning that

$$A(B+C) = AB + AC$$

and

$$(A+B)C = AC + BC$$

for any matrices A, B, and C that can be added and multiplied.

A square matrix is a matrix that has the same number of rows and columns, denoted by n.

$$A_{n \times n}$$

A square matrix A is called diagonal if $A_{ij} = 0$ for all i and j such that i≠j.

A diagonal matrix can be written as

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

, where a_{ii} are the diagonal elements.

A diagonal matrix is called identity if A_{ij} =1 for all i and j such that i=j, and 0 otherwise.

An identity matrix, denoted by I, can be written as

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}$$

An identity matrix has the property that

$$AI = A \quad \text{and} \quad IA = A$$

for any matrix A of the same size.

The trace of a matrix is defined as the sum of the principal diagonal elements of a square matrix. It is usually represented as tr(A), where A is any square matrix of order "n × n." For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

In index notation, $tr(A) = a_{11} + a_{12} + a_{13} + \dots$

$$= \sum_{i=1}^{N} a_{ii}$$

then tr(A) = 1 + 4 = 5.

The trace of a matrix has some important properties, such as:

- The trace of a matrix is equal to the sum of its eigenvalues (counted with multiplicities).
- The trace of a matrix is invariant under similarity transformations, that is, if A and B are similar matrices, then tr(A) = tr(B).
- The trace of a matrix is equal to the trace of its transpose, that is, tr(A) = tr(A^T).
- The trace of a matrix is linear with respect to matrix addition and scalar multiplication, that is, tr(A + B) = tr(A) + tr(B) and tr(kA) = k tr(A), where k is any scalar.
- The trace of a matrix is cyclic with respect to matrix multiplication, that is, tr(AB) = tr(BA) for any matrices A and B such that AB and BA are defined.

$$tr(ABC) = tr(BCA) = tr(CAB)$$

Some examples of matrices and their traces are:

- The trace of an identity matrix of order "n × n" is n, that is, tr(I_n) = n.
- The trace of a zero or null matrix of any order is zero, that is, tr(O) = 0.

An idempotent matrix is a square matrix that does not change when multiplied by itself. In other words, if A is an idempotent matrix, then $A^2 = A$. For example, the identity matrix I is idempotent, since $I^2 = I$. Idempotent matrices have some interesting properties, such as:

- Their eigenvalues are either 0 or 1.
- Their determinant is either 0 or 1.
- Their trace is equal to their rank.
- They are always diagonalizable.

A nilpotent matrix is a square matrix that becomes zero when raised to some positive integer power. In other words, if A is a nilpotent matrix, then there exists a positive integer k such that $A^k = 0$. For example, the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is nilpotent, since $A^3 = 0$. k is called the index. Nilpotent matrices have some interesting properties, such as:

- Their determinant is always zero.
- Their trace is always zero.
- Their eigenvalues are always zero.
- They are not invertible, unless they are the zero matrix.

An involutory matrix is a square matrix that is its own inverse, meaning that multiplying it by itself gives the identity matrix of the same order. For example, the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is an involutory matrix, because

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Some properties of involutory matrices are:

- The determinant of an involutory matrix is always either +1 or -1.
- The eigenvalues of an involutory matrix are always either +1 or -1.
- A symmetric involutory matrix is orthogonal, and vice versa.

A symmetric matrix is a square matrix that is equal to its transpose, meaning that the elements on the opposite sides of the main diagonal are the same. For example, the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

is a symmetric matrix, because it is equal to its transpose, which is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Some of the properties of symmetric matrices are:

- The sum and difference of two symmetric matrices are also symmetric.
- The product of two symmetric matrices is symmetric if and only if they commute, that is, if AB = BA.
- The determinant of a symmetric matrix is equal to the determinant of its transpose, and it is always a real number.
- The inverse of a symmetric matrix is also symmetric, if it exists.
- The eigenvalues of a symmetric matrix are always real, and its eigenvectors are orthogonal and real.

Symmetric matrices have many applications in mathematics, physics, engineering, and computer science. For example, they can be used to represent covariance matrices, quadratic forms, inner products, orthogonal transformations, and more.

A skew-symmetric matrix is a square matrix that is equal to the negative of its transpose. In other words, if A is a skew-symmetric matrix, then A^T = -A. This means that the elements on the opposite sides of the main diagonal are the opposite of each other, and the elements on the main diagonal are all zero. For example, the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}$$

is a skew-symmetric matrix, because

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

Some of the properties of skew-symmetric matrices are:

- The sum and difference of two skew-symmetric matrices are also skew-symmetric.
- The product of two skew-symmetric matrices is symmetric if and only if they commute, that is, if AB = BA.
- The determinant of a skew-symmetric matrix is always zero if the order of the matrix is odd, and always a non-positive perfect square if the order of the matrix is even.
- The inverse of a skew-symmetric matrix is also skew-symmetric, if it exists.
- The eigenvalues of a skew-symmetric matrix are always either zero or purely imaginary, and its eigenvectors are orthogonal and complex.

Skew-symmetric matrices have many applications in mathematics, physics, engineering, and computer science. For example, they can be used to represent cross products, angular velocities, antisymmetric tensors, exterior derivatives, and more.

A hermitian matrix is a square matrix that is equal to its conjugate transpose, that is, a matrix A such that $A^{+} = A$, where A^{+} is the matrix obtained by taking the complex conjugate of each element and then transposing the matrix. A hermitian matrix has some important properties, such as:

- The diagonal elements of a hermitian matrix are always real numbers.
- The eigenvalues of a hermitian matrix are always real numbers, and the eigenvectors corresponding to distinct eigenvalues are orthogonal.
- A hermitian matrix can be diagonalized by a unitary matrix, that is, there exists a matrix U such that, $U^{*} A U = D$ where D is a diagonal matrix with the eigenvalues of A on the diagonal.
- The determinant, trace, and inverse of a hermitian matrix are also real numbers.
- The sum and product of two hermitian matrices are hermitian if and only if they commute, that is, AB=BA.

Some examples of hermitian matrices are:

- Any real symmetric matrix, such as $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ or $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$
- Any complex matrix with real diagonal elements and conjugate symmetric off-diagonal elements, such as

$$\begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 2+i & -1 \\ 2-i & 4 & 3-i \\ -1 & 3+i & 5 \end{bmatrix}$$

A skew-hermitian matrix is a square matrix of complex numbers that is equal to the negative of its conjugate transpose. That is, if A is a skew-hermitian matrix, then $A^{\dagger}=-A$, where $A^{\dagger}$ is the matrix obtained by taking the complex conjugate of each element and then transposing the matrix. A skew-hermitian matrix has some properties opposite to those of hermitian matrices, such as:

- The diagonal elements of a skew-hermitian matrix are always zero or purely imaginary numbers.
- The eigenvalues of a skew-hermitian matrix are always purely imaginary numbers, and the eigenvectors corresponding to distinct eigenvalues are orthogonal.
- A skew-hermitian matrix can be diagonalized by a unitary matrix, that is, there exists a matrix U such that $U^{\dagger}AU=D$, where D is a diagonal matrix with the eigenvalues of A on the diagonal.
- The determinant, trace, of a skew-hermitian matrix are also purely imaginary numbers.
- The sum and product of two skew-hermitian matrices are skew-hermitian if and only if they commute, that is, AB=BA.

Some examples of skew-hermitian matrices are:

$$\begin{bmatrix} 0 & i & 1+i \\ -i & 0 & 2 \\ -1-i & -2 & 0 \end{bmatrix} \quad \begin{bmatrix} 2i & 3+4i \\ -3-4i & -2i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Skew-hermitian matrices can be used to represent the Lie algebra of the unitary group, which is important in quantum mechanics and differential geometry.

COUNTING UNIQUE ELEMENTS OF MATRICES.

(i) Symmetric Matrix : $A_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

For off-diagonal entries

$$a_{ij} = a_{ji} \quad \text{for} \quad i \neq j$$

Thus, unique off diagonal entries.

$= \frac{1}{2}$ (total diagonal entries.).

Now, total off-diagonal entries = total matrix entries − no. of diagonal entries

$= n^2 - n$

$\therefore$ unique off-diagonal entries $= \dfrac{n^2 - n}{2}$

For diagonal entries.

there are n unique diagonal entries a_{ii}

total unique entries = unique off-diagonal entries + unique diagonal entries

$$= \frac{n^2 - n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

(ii) Skew-Symmetric Matrix

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

For off-diagonal entries

$$a_{ij} = -a_{ji} \quad \text{for} \quad i \neq j$$

Thus, unique off diagonal entries.

$= \frac{1}{2}$ (total diagonal entries.).

Now, total off-diagonal entries = total matrix entries − no. of diagonal entries

$= n^2 - n$

$\therefore$ unique off-diagonal entries $= \dfrac{n^2 - n}{2}$

For diagonal entries.

there are 0 unique diagonal entries ($a_{ii} = 0$)

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total unique entries = unique off-diagonal entries + unique diagonal entries

$$= \frac{n^2 - n}{2} + 0 = \frac{n(n-1)}{2}$$

(iii) Hermitian - matrix ($A_{n\times n}$)

$$\begin{bmatrix} a_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} \\ a_{12}-ib_{12} & a_{22} & a_{23}+ib_{23} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_{33} \end{bmatrix}$$

For diagonal entries.

all diagonal entries of a Hermitian - matrix are real.

there are n diagonal entries.

So, there are total n real numbers in n diagonal entries.

For off diagonal entries.

off - diagonal entries of a Hermitian matrix are complex.

there are a total of $n^2 - n$ diagonal entries.

Each entry has two real numbers a_{ij} and b_{ij}.

Hence, there are a total of $2(n^2-n)$ real numbers in off - diagonal entries.

But off-diagonal entries are complex conjugates of each other. Hence, there are

$\frac{2(n^2-n)}{2}$ unique real numbers in off-diagonal entries.

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total unique real numbers $= n + \frac{2(n^2-n)}{2}$

$= n + n^2 - n$

$= n^2$

(iii) Skew-Hermitian - matrix $(A_{n\times n})$

$$\begin{bmatrix} ib_{11} & a_{12}+ib_{12} & a_{13}+ib_{13} \\ -a_{12}+ib_{12} & ib_{22} & a_{23}+ib_{23} \\ -a_{13}+ib_{13} & -a_{23}+ib_{23} & ib_{33} \end{bmatrix}$$

For diagonal entries.

all diagonal entries of a Hermitian - matrix are purely imaginary.

there are n diagonal entries.

So, there are total n real numbers in n diagonal entries.

For off diagonal entries.

off - diagonal entries of a Hermitian matrix are complex.

there are a total of n^2-n diagonal entries.

Each entry has two real numbers a_{ij} and b_{ij}.

Hence, there are a total of $2(n^2-n)$ real numbers in off - diagonal entries.

But off-diagonal entries are negative complex conjugates of each other. Hence, there are

$\frac{2(n^2-n)}{2}$ unique real numbers in off-diagonal entries.

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total unique real numbers $= n + \frac{2(n^2-n)}{2}$

$= n + n^2 - n$

$= n^2$

An orthogonal matrix is a square matrix whose transpose is equal to its inverse, meaning that multiplying it by itself gives the identity matrix of the same order. For example, the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is an orthogonal matrix, because

$$\because \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Some properties of orthogonal matrices are:

- The determinant of an orthogonal matrix is always either +1 or -1.
- The eigenvalues of an orthogonal matrix are always either +1 or -1, and its eigenvectors are orthogonal and real.
- An orthogonal matrix is symmetric if and only if it is diagonal.
- An orthogonal matrix is orthogonal, and vice versa.

Orthogonal matrices have many applications in mathematics, physics, engineering, and computer science. For example, they can be used to represent rotations, reflections, and orthogonal transformations. They also preserve the length and angle of vectors, which makes them useful for solving linear equations and optimization problems.

1 : Prove that the determinant of an orthogonal matrix is either +1 or −1.

Suppose, A is an orthogonal matrix,

then $AA^T = I$.

$\Rightarrow \det(AA^T) = \det(I)$

$\Rightarrow \det(A) \cdot \det(A^T) = 1$

Since $\det(A) = \det(A^T)$.

$\therefore \det(A) \cdot \det(A^T) = 1$

$\Rightarrow \{\det(A)\}^2 = 1$

$\Rightarrow \det(A) = \pm 1$

A unitary matrix is a **square matrix** whose entries are **complex numbers**.

It possesses a remarkable property: when multiplied by its **conjugate transpose**, it equals the **identity matrix**.

In other words, if U is a unitary matrix, then either of the following conditions holds:

$U^{+} = U^{-1}$ (where (U^{+}) is the conjugate transpose of U).

($U^{+}U = I = UU^{+}$), where I represents the identity matrix.

Properties:

Preservation of Norms: Unitary matrices play a crucial role in **quantum mechanics** because they preserve **norms**. This preservation ensures that **probability amplitudes** remain consistent.

Diagonalizability: Every unitary matrix is **diagonalizable**. This implies that it can be transformed into a diagonal matrix through a unitary similarity transformation.

Determinant: The determinant of a unitary matrix has a **modulus of 1**. In other words, (|det(U)| = 1), placing it on the unit circle in the complex plane.

Orthogonality: The eigenspaces of a unitary matrix are **orthogonal**.

2. Prove that the determinat of an unitary matrix is $1, -1, i$ or $-i$.

Suppose A is an unitary matrix.

then $AA^\dagger = I$

or $A(A^T)^* = I$

Now. $\det\left(A(A^T)^*\right) = \det(I)$

$\Rightarrow \det(A)\det\left((A^T)^*\right) = 1$

$\Rightarrow \det(A)\left(\det(A^T)\right)^* = 1$

$\Rightarrow \det(A)\left(\det(A)\right)^* = 1$

$\Rightarrow |\det(A)|^2 = 1$

So, $\det(A)$ must be $+1, -1, +i$ or $-i$

3. Prove that the eigenvalues of a Hermitian matrix are real.

Let us suppose X is an eigenvector corresponding to eigenvalue λ for a Hermitian matrix A.

$$AX = \lambda X$$

$$\Rightarrow X^{+}AX = \lambda X^{+}X \rightarrow (1)$$

$$AX = \lambda X$$

$$\Rightarrow (AX)^{+} = (\lambda X)^{+}$$

$$\Rightarrow X^{+}A^{+} = \lambda^{*} X^{+}$$

$$\Rightarrow X^{+}A^{+}X = \lambda^{*} X^{+}X$$

$$\Rightarrow X^{+}AX = \lambda^{*} X^{+}X \quad \left(\because A = A^{+} \text{ as } A \text{ is hermitian} \right) \rightarrow (2)$$

From (1) and (2)

$$\lambda X^{+}X = \lambda^{*} X^{+}X$$

$\underbrace{\lambda}_{a+ib} = \underbrace{\lambda^{*}}_{a-ib}$ holds true only when λ is real.

Hence, eigenvalues of a Hermitian matrix are real.

Q: Prove that the eigenvectors corresponding to different eigenvalues of a Hermitian matrix are orthogonal.

Suppose A is a Hermitian matrix.

λ_i and λ_j are two different eigenvalues.

x_i is an eigenvector corresponding to λ_i

x_j is an eigenvector corresponding to λ_j

$$A x_i = \lambda_i x_i$$

$$\Rightarrow (A x_i)^\dagger = (\lambda_i x_i)^\dagger$$

$$\Rightarrow x_i^\dagger A = \lambda_i^* x_i^\dagger$$

$$\Rightarrow x_i^\dagger A x_j = \lambda_i^* x_i^\dagger x_j$$

$$\Rightarrow x_i^\dagger A x_j = \lambda_i x_i^\dagger x_j \qquad \left[\lambda_i = \lambda_i^* \text{ since eigenvalues of Hermitian matrices are real} \right] \rightarrow (1)$$

Similarly, $A x_j = \lambda_j x_j$

$$\Rightarrow x_i^\dagger A x_j = \lambda_j x_i^\dagger x_j \rightarrow (2)$$

From (1) and (2),

$$\lambda_i x_i^\dagger x_j = \lambda_j x_i^\dagger x_j$$

$$\Rightarrow (\lambda_i - \lambda_j)\, x_i^+ x_j = 0.$$

Since λ_i and λ_j are different,

Hence $\lambda_i - \lambda_j \neq 0$.

So, $x_i^+ x_j$ must be 0.

Similarly we can show that

$$x_j^+ x_i = 0 .$$

This proves that x_i and x_j are orthogonal to each other.

Caley - Hamilton Theorem.

Every square matrix satisfies its own characteristic equation - $|A - \lambda I| = 0$

For eg :- $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

According to the Caley - Hamilton Theorem, the matrix A shall satisfy this equation

ie. $A^2 - 4A - 5 = 0$

Let us verify this.

$A^2 - 4A - 5$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Finding A^{-1} using the Caley-Hamilton theorem

The characteristic equation is -

$$\lambda^2 - 4\lambda - 5 = 0$$

According to the Caley-Hamilton Theorem,

$$A^2 - 4A - 5I = 0$$

$$\Rightarrow AAA^{-1} - 4AA^{-1} - 5IA^{-1} = 0$$

$$\Rightarrow A - 4I - 5A^{-1} = 0$$

$$\Rightarrow \quad 5A^{-1} = A - 4I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 & 4/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3/5 & 4/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 + 8/5 & 4/5 - 4/5 \\ -6/5 + 6/5 & 8/5 - 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 5/5 & 0 \\ 0 & 5/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

DIAGONALIZATION OF MATRIX A.

$$A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

Step I : Find the eigenvalues using the characteristic eq^n.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -3 & -3 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[(-2-\lambda)(2-\lambda) - 1(-3)\right]$$
$$- (-3)\left[3(2-\lambda) - (-1)(-3)\right]$$
$$+ (-3)\left[3 - (-1)(-2-\lambda)\right] = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4 + 3) + 3(3 - 3\lambda) - 3(1-\lambda) = 0$$
$$\Rightarrow (4-\lambda)(\lambda^2 - 1) - 9(\lambda - 1) + 3(\lambda - 1) = 0$$
$$\Rightarrow (\lambda - 1)\left((4-\lambda)(\lambda+1) - 9 + 3\right) = 0$$
$$\Rightarrow (\lambda - 1)(4\lambda - \lambda^2 + 4 - \lambda - 6) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-3\lambda+2)=0$$

$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-2)=0$$

Thus, the eigenvalues are 1, 1, 2.

Step II : Find the eigenvectors corresponding to the three eigenvalues.

For $\lambda = 1$,

$$(A - I\lambda)x = 0$$

$$\Rightarrow \begin{bmatrix} 4-1 & -3 & -3 \\ 3 & -2-1 & -3 \\ -1 & 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 3 & -3 & -3 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow 3x - 3y - 3z = 0$$

$$\Rightarrow x - y - z = 0$$

$$\Rightarrow x = y + z.$$

Two eigenvectors corresponding to $\lambda = 1$ are.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 2$

$$(A - I\lambda)X = 0$$

$$\Rightarrow \begin{bmatrix} 4-2 & -3 & -3 \\ 3 & -2-2 & -3 \\ -1 & 1 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & -3 \\ 3 & -4 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$2x - 3y - 3z = 0 \rightarrow (i)$$

$$3x - 4y - 3z = 0 \rightarrow (ii)$$

$$-x + y = 0 \rightarrow (iii)$$

$$\Rightarrow x = y.$$

Putting $x = y$ in (ii)

$$3x - 4x - 3z = 0$$

$$\Rightarrow -x - 3z = 0$$

$$\Rightarrow x = -3z$$

Eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$

Step III : Adjoin all eigenvectors so as to form a full rank matrix P

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

Step IV : Find P^{-1}

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} = (-1)(-3) = 3$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = (-1)(-1) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -1 - 3 = -4$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (-1)(-1) = 1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3-3 = 0$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = (-1)\,3 = -3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\det P = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 1(3-3) - 0(3-0) + (-1)(1-0)$$

$$= -1.$$

So, $$P^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -4 & 1 \\ 0 & -3 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Step V : Find the diagonal matrix $D = P^{-1}AP$

$$D = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4+0-3 & 4-3+0 & 12-9+3 \\ 3+0-3 & 3-2+0 & 9-6+3 \\ -1+0+2 & -1+1+0 & -3+3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 6 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1+0 & 1-1+0 & 6-6+0 \\ -3+0+3 & -3+4+0 & -18+24-6 \\ 1+0-1 & 1-1+0 & 6-6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This is the required diagonal matrix.

Note the following :-

1. the diagonal elements of D are the eigenvalues of A.

2. A and D are called "similar matrix" because

   (i) trace of A = trace of D.

   (ii) $\det(A) = \det(D)$

   (iii) eigenvalues of A = eigenvalues of D

Two matrices with equal trace, determinant and eigenvalues are called similar matrices.

the transformation is called similarity transformation.