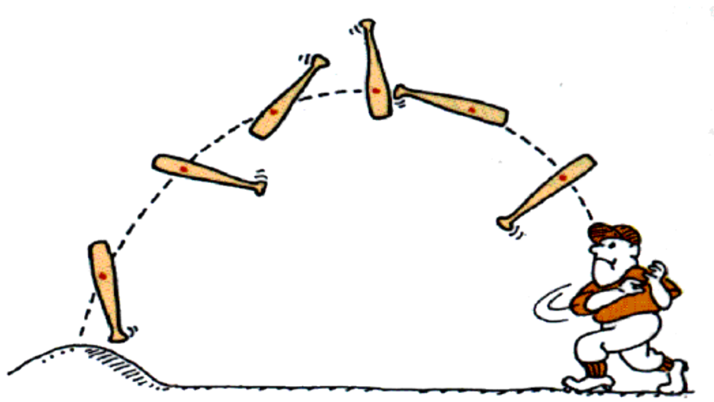
Unit IV - Dynamics of Rigid Bodies

The dynamics of a rigid body deals with the study of the motion of an object that maintains its shape and does not deform under the influence of external forces. Unlike a system of particles, which can have individual particles with different positions and velocities, a rigid body is a single object with a fixed configuration. The dynamics of a rigid body is a fundamental topic in mechanics and has numerous applications in engineering, robotics, aerospace, and other fields. It provides a framework for analyzing the motion, stability, and behaviour of objects that can be approximated as rigid bodies.

The dynamics of a rigid body is governed by Newton's laws of motion, similar to the dynamics of particles. Newton's laws can be extended to describe the translational and rotational motion of a rigid body. They relate the net force and net torque acting on a rigid body to its linear acceleration and angular acceleration, respectively. Dynamics of a rigid body involves the analysis of moments and torques. A moment is a measure of the tendency of a force to cause rotation about a particular axis. Torque is the rotational equivalent of force and is responsible for producing angular acceleration in a rigid body.

A rigid body can undergo translational motion, where the entire body moves in a straight line without any rotation. In this case, all points of the rigid body have the same velocity and follow parallel paths. A rigid body can also undergo rotational motion, where it spins about a fixed axis. Each point of the body moves in a circular path centred on the axis of rotation. The motion of a rotating rigid body can be described in terms of angular displacement, angular velocity, and angular acceleration. In general, a rigid body can undergo both translation and rotation simultaneously. This combined motion is known as general motion. It involves a combination of linear motion of the centre of mass and rotational motion about an axis passing through the centre of mass. The moment of inertia quantifies the resistance of a rigid body to rotational motion. It depends on the distribution of mass within the body relative to the axis of rotation. The larger the moment of inertia, the greater the resistance to changes in rotational motion. Conservation laws, such as the conservation of linear momentum and the conservation of angular momentum, play a crucial role in the dynamics of a rigid body.



Rigid body motion – Translational and Rotational motion

A rigid body is defined as a solid body in which the particles are compactly arranged so that the inter-particle distance is small and fixed and is not disturbed by any external forces applied. Such a body thus preserves its shape or configuration intact and does not bend, stretch or vibrate when in motion.

A rigid body may either move bodily, i.e., as a whole, in any direction, or it may rotate in two or three dimensions. In the former case, it is said to execute a translatory motion and in the latter, a rotational motion. A body may also execute both translational and rotational motions simultaneously. We can imagine the translational motion of a rigid body to be that of one single point, i.e., its centre of mass, where the whole mass of the body is supposed to be concentrated there.

The rotational motion of a rigid body can be about a line or axis fixed in the reference frame being referred to as the axis of rotation of the body. A given point in the body thus moves in a plane perpendicular to this axis of rotation, called the plane of rotation and the rotation is referred to as rotation in two dimensions.

Let us consider a rigid body free to rotate about the axis OZ through O, with the axis fixed in an inertial reference frame. All the particles of the body describe circles about the axis of rotation (OZ) in a plane perpendicular to it and have their centres lying on it, with their radii equal to their respective perpendicular distances from it and the radius of each obviously sweeps through the same angle $\delta\theta$, say, in the same time δt.

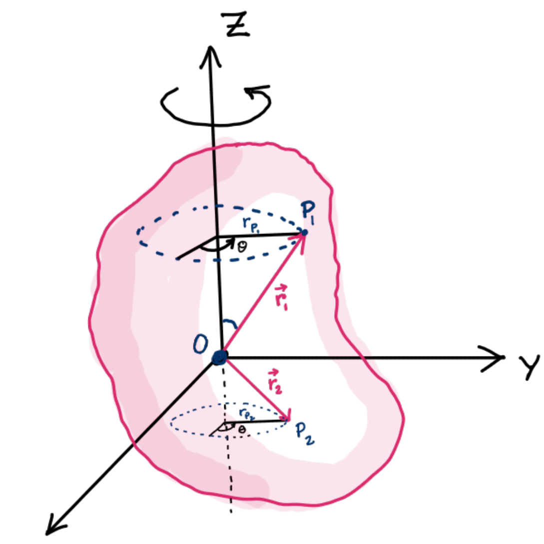


fig 1

Thus, for example, particles $P1$ and $P2$, with position vector $\vec{r_1}$ and $\vec{r_2}$ with respect to origin O and distance r_{p1} and r_{p2} from the axis, describe circles of radii r_{p1} and r_{p2} with their centres respectively.

The angular velocity of both $P1$ and $P2$, and in fact of all the particles, and hence of the body, as a whole, is thus the same, viz., $\vec{\omega} = \frac{\delta\vec{\theta}}{\delta t}$, which, in the limit δt → 0 gives $\vec{\omega} = \frac{d\vec{\theta}}{dt}$, the direction of which (in accordance with the right hand thumb rule) is along the axis of rotation (or the Z-axis), upward, if the rotation of the body is anticlockwise about the axis. This is indicated in the figure by the thick arrow, the length of which is proportional to the magnitude of ω. It merely represents conventionally the angular velocity of the rotational motion that is taking place in the plane perpendicular to the x — y plane, in this case shown.

The angular acceleration of the body is given by $\vec{a} = \frac{d\vec{\omega}}{dt}$. The linear speed of a particle P is given by $v = \omega r$ (i.e., its angular velocity × radius of the circle in which it rotates), where v and ω

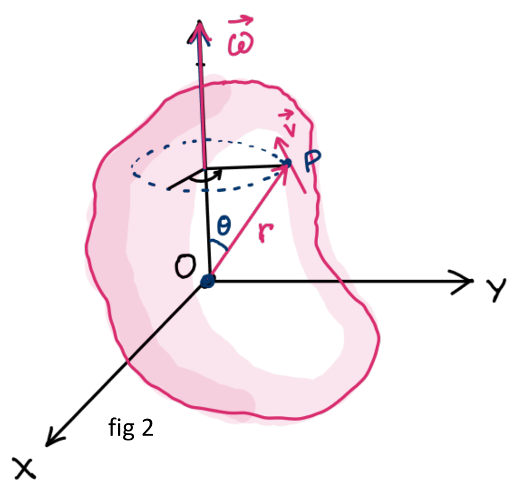


fig 2

are respectively the magnitudes of the linear speed and the angular velocity.

If $\vec{r}$ is the position vector of a particle P of a rigid body (as shown in fig. 2) with respect to the origin O, it rotates about the axis in a circle of radius $r_p = r \sin \angle POA = r \sin \theta$, where θ is the angle that the position vector $\vec{r}$ makes with the axis. So that, $v = \omega r \sin\theta$.

And therefore, linear velocity vector $\vec{v} = \vec{\omega} \times \vec{r}$, points in a direction is tangential to the circular path at P and perpendicular to the position vector $\vec{r}$, as shown.

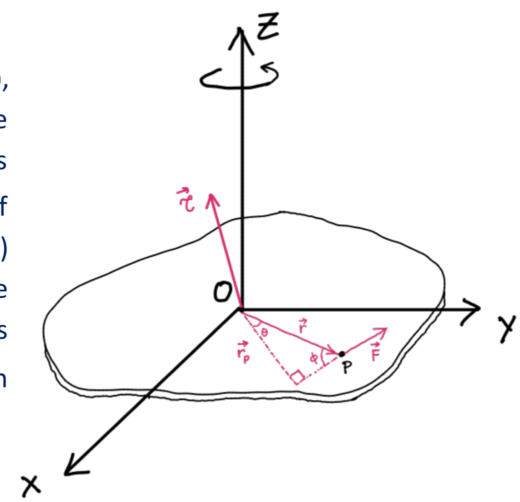
In this chapter, we will restrict ourselves with only this plane rotational motion of a rigid body and the study of its relationship with the properties of the body or the causes of its rotation, i.e., the rotational dynamics of the rigid body.

Some Concepts related to Rotational motion

1. Torque

In case of the translational motion, we need to know only the magnitude of the force applied for linear acceleration. In case of the rotational motion, for angular acceleration, we must know the actual point of application of the force and the way it is directed, or, in other words, the moment of the force or the 'Torque'.

If $\vec{F}$ is the force acting on a particle P of a rigid body (shown in fig3), the moment of the force about the origin O in an inertial reference frame, or the torque τ acting on the particle, with respect to O, is $\vec{F} \times \overrightarrow{r_P}$, where $\overrightarrow{r_P}$ is the perpendicular distance between the line of action of the force and the origin O (or the axis passing through O) also known as the moment arm. If $\vec{r}$ is the position vector of the particle at P with respect to O, we have $r_p = r \sin \varphi$, where ϕ is the angle between $\vec{F}$ and $\vec{r}$. The magnitude of the torque acting on the particle is $Fr \sin \varphi$. Vectorially,



$$\vec{\tau} = \vec{r} \times \vec{F}$$

The direction of the torque is as shown, given by the right-handed thumb rule, according to which if we curl the fingers of our right hand in the direction in which $\vec{r}$ must be swung to move the position of $\vec{F}$, through the smaller angle between them, the extended thumb gives the direction or the sense of the vector $\vec{\tau}$. If $\vec{r}$ = 0, i.e., the point P lies at the origin O, the torque is zero. In other words, the torque about the origin itself is zero. From the relation, it is clear that only the component of $\vec{F}$ perpendicular to $\vec{r}$ is responsible for the torque.

For a rigid body consisting of a large number of particles, the torque can be written as,

$$\vec{\tau} = \sum(\vec{r} \times \overrightarrow{F})$$

The torque has the same dimensions as work (both being force times distance), viz., ML^2T^{-2}. The two are, however, entirely different physical quantities; whereas work is a scalar quantity, torque is a vector.

2. Angular momentum

Angular momentum, in rotational motion, is the analogue of linear momentum, in translational motion, and is defined as the moment of linear momentum.

The linear momentum for a particle of mass m, moving with velocity $\vec{v}$ is given by $\vec{p} = m\vec{v}$ and for a system of particles or a rigid body, by $\vec{P} = M\overrightarrow{v_{c.m.}}$, where M is the mass of the whole system or body and $\overrightarrow{v_{c.m.}}$ is the velocity of its centre of mass. Therefore, considering a particle of a rigid body, free to turn about the fixed axis OZ through O, if its linear momentum $\vec{p}$ be directed as shown, similar to the point of action of the force $\vec{F}$, such that it makes ∠ϕ with its position vector $\vec{r}$ (with respect to the origin O) in an inertial frame of reference, its angular momentum will be given by

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

It is a vector quantity whose magnitude is L = pr sin ϕ, and its direction is given by the right-hand thumb rule for the vector product of the two vectors as in the case of torque discussed above. It is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$. From the relation, it is clear that, similar to the case of torque, only the component of $\vec{p}$ perpendicular to $\vec{r}$ is responsible for the angular momentum.

For a rigid body consisting of a large number of particles, the angular momentum can be written as,

$$\vec{L} = \sum(\vec{r} \times \overrightarrow{p}) = \sum(m\vec{r} \times \overrightarrow{v}) .$$

Differentiating the above relation of $\vec{L}$, we get

$$\frac{d\vec{L}}{dt} = \sum\vec{r} \times \frac{d}{dt}(m\vec{v}) = \sum(\vec{r} \times \overrightarrow{F}) = \vec{\tau} ,$$

where $\sum(\vec{r} \times \overrightarrow{F})$ is the sum of the torques, or the total torque, due to all the external forces acting on the system or the rigid body. The reason why we explicitly use the word external is that internal forces all form collinear pairs of opposite forces (of action and reaction in accordance with Newton's third law of motion), having equal and opposite moments about the given point, so that their sum is zero and they produce no effect. This relation is the fundamental equation of motion of a rigid body about a fixed axis through a given point O and applies irrespective of whether the particles constituting the system are in motion relative to each other or in fixed spatial relationship with each other, as in a rigid body.

In the absence of external torque, $\frac{d\vec{L}}{dt} = \vec{\tau} = 0$ which shows $\vec{L}$ is constant. Thus, in this case of rotational motion, the law of conservation of angular momentum holds good. According to this rule, in the absence of any external forces, the angular momentum of the rigid body remains constant.

3. Moment of Inertia

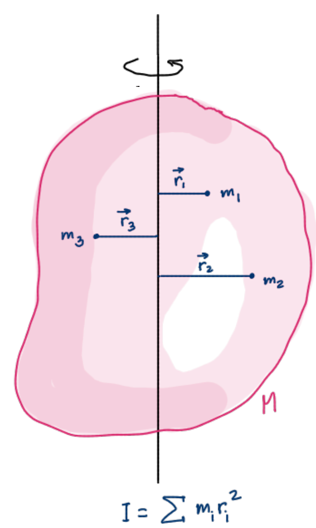
The inability of a body to change by itself its state of rest or uniform motion along a straight line (Newton's first law of motion) is an inherent property of matter and is called inertia. The greater the mass of the body, the greater the resistance offered by it to any change in its state of rest or linear motion. Here mass is taken to be a measure of inertia for linear or translatory motion. In exactly the same manner, a body free to rotate about an axis opposes any change in its state of rest or uniform rotation. In other words, it possesses inertia for rotational motion, i.e., it opposes the torque applied to it to change its state of rotation. Hence, the name moment of inertia given to it. It is also known as rotational inertia.

The moment of inertia about a given axis plays the same part in rotational motion about that axis as the mass of a body does in translational motion. In other words, moment of inertia, in rotational motion, is the analogue of mass in linear or translational motion.

Mathematically, the moment of inertia of a particle about a given axis of rotation is defined as the product of its mass and the square of its distance from the axis. Thus, moment of inertia of a particle, is

$$I = mr^2$$

where r is the perpendicular distance of the particle from rotational axis. Similarly, moment of inertia of a rigid body made up of a number of particles (discrete distribution) can be expressed as 'the moment of inertia of a rigid body about a given axis of rotation as the sum of the products of the masses of the various particles of the body and the squares of their respective distances from the axis'.



Thus, if r_1, r_2, r_3 etc. be the perpendicular distances, from the axis, of particles of respective masses m_1, m_2, m_3, etc., (as shown in the fig. 4) we have

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... = \sum mr^2 = MK^2,$$

where M is the total mass of the body, $M = \sum m$ and K is the effective distance of its particles from the axis, called its radius of gyration about the axis of rotation.

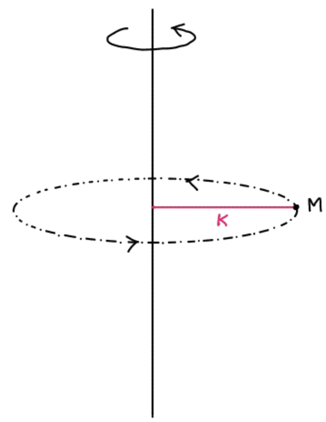
In the case of a body which does not consist of separate, discrete particles but has a continuous and homogeneous distribution of matter in it, the summation becomes an integration, the integral being taken over the entire body. Thus, $I = \int mr^2 dm = MK^2$, where dm is the mass of an infinitesimally small element of the body at distance r from the axis.

4. Radius of Gyration

Radius of gyration of a body about a given axis is the perpendicular distance of a point from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass.

When square of radius of gyration (K) is multiplied with the mass of the body (M), it gives the moment of inertia (I) of the body about the given axis.

$I = MK^2$ or $K = \sqrt{\frac{I}{M}}$



From the formula of discrete distribution

$I = mr_1^2 + mr_2^2 + mr_3^2 + ... + mr_n^2 = I_1 + I_2 + I_3 + ...$

Thus, the total moment of Inertia of a number of particles about a given axis is equal to the sum of the moments of inertia of the individual particles about that axis, where I is the total moment of inertia of the bodies and I_1, I_2, I_3 etc. their individual moments of inertia about the given axis. Therefore, if a body has a moment of inertia about a given axis equal to I, and a portion of it, having a moment of inertia I' about the same axis be removed from it, the moments of inertia of the remaining part of the body about the given axis will be $I - I'$.

If $m_1 = m_2 = m_3 = ... = m$, then, $I = m(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2)$

From the definition of Radius of gyration,

$$I = MK^2 = m(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2)$$

$$\Longrightarrow mnK^2 = m(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2) \qquad \text{[As M = mn]}$$

$$\therefore K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}{n}}$$

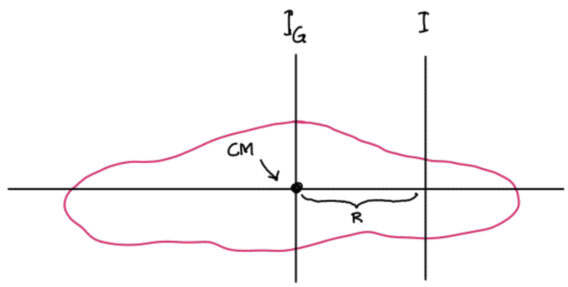
Hence radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

Radius of gyration depends on shape and size of the body, position and configuration of the axis of rotation, distribution of mass of the body with respect to the axis of rotation but does not depend on the mass of body. Its dimension is $[M^0 L^1 T^0]$. Its S.I. unit is meter. It is a scalar quantity. Talking about its significance, through this concept a real body (particularly irregular) is replaced by a point mass for dealing its rotational motion. Clearly, therefore, a change in the position or inclination of the axis of rotation of a body will bring about a change in the relative distances of its particles and hence in their effective distance or the radius of gyration of the body about the axis.

5. Theorems of Moment of Inertia

5.1 Theorem of Parallel Axis or Steiner's theorem

This theorem states that '*the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass, plus the product of the mass of the body and the square of the distance between the two axes.*'

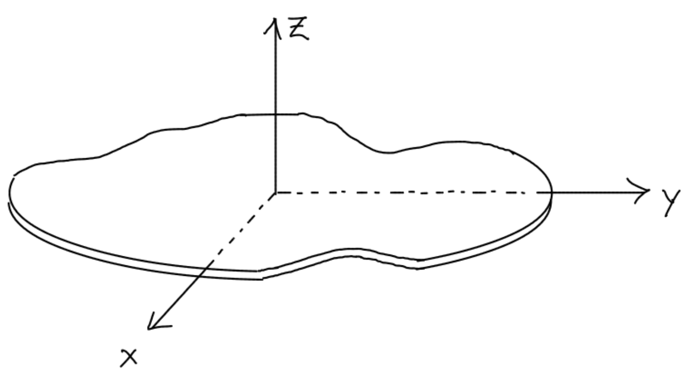


If I is the moment of inertia of a body about a given axis and I_G be the moment of inertia of the body about an axis parallel to given axis and passing through centre of mass of the body of mass M and R is the perpendicular distance between the two axes, then

$$I = I_G + MR^2$$

5.2 Theorem of Perpendicular Axis

This theorem states that '*the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two mutually perpendicular axes, in its own plane, and intersecting each other at the point where the perpendicular axis passes through it*'.



If I_x, I_y and I_z are the moments of inertia about the three axes, i.e., X-,Y- and Z- axes respectively, mutually perpendicular to each other, then,

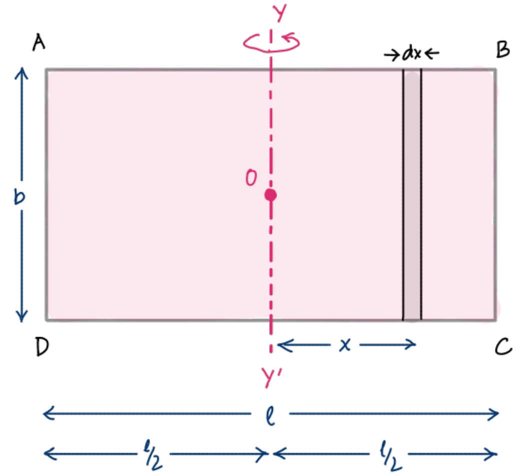
$$I_z = I_x + I_y .$$

Moment of Inertia (M.I.) of various geometrical bodies

1. Moment of inertia of rectangular lamina

(i) about an axis through its centre and parallel to one side.

Let ABCD be a rectangular lamina, of length l, breadth b and mass M and let YOY' be the axis through its centre O and parallel to the side AD or BC about which its moment of inertia is to be determined.



Let us consider an elemental rectangular strip of the lamina, parallel to, and at a distance x from the axis. The area of the element $= dx \times b$.

And, since the mass per unit area of the lamina $= M/(l \times b)$, so the mass of the elemental strip will be $\frac{M}{l \times b} \times dx \times b = \frac{M}{l} \times dx$.

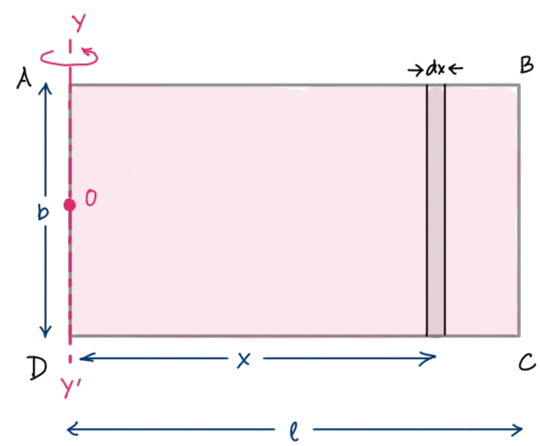
Therefore, Moment of Inertia of the element about the axis YOY' $= \frac{M}{l}.dx.x^2$

The Moment of Inertia, (I) of the whole rectangular lamina is thus given by twice the integral of the above expression between the limits *x = 0 and x = l/2*.

Therefore, $$I = 2\int_0^{l/2} \frac{M}{l} x^2 dx = \frac{2M}{l}\int_0^{l/2} x^2 dx = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{\frac{l}{2}} = \frac{2M}{l}.\frac{l^3}{24} = \frac{Ml^2}{12}$$

(ii) about an axis through one side

In this case, since the axis coincides with AD or BC, we integrate the expression for the M.I. of the element of length dx at distance x from the axis, i.e.,



$(M/l)\ dx.x^2$,

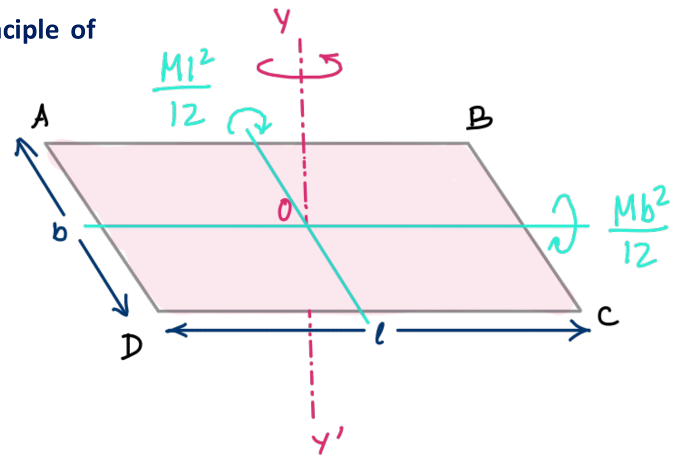
between the limits $x = 0$ at AD and $x = l$ at BC. So that M.I. of the lamina about side AD or BC is given by

$$I = \int_0^l \frac{M}{l} x^2 dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{Ml^2}{3}$$

(iii) About an axis passing through its centre and perpendicular to its plane

This case can be obtained by an applying the **principle of perpendicular axes** to case (i) above, according to which, M.I. of the lamina about an axis through O and perpendicular to its plane = M.I. of the lamina about an axis through O parallel to b + M.I. of the lamina about an axis through O parallel to l, i.e.,

$$I = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M(l^2+b^2)}{12}$$



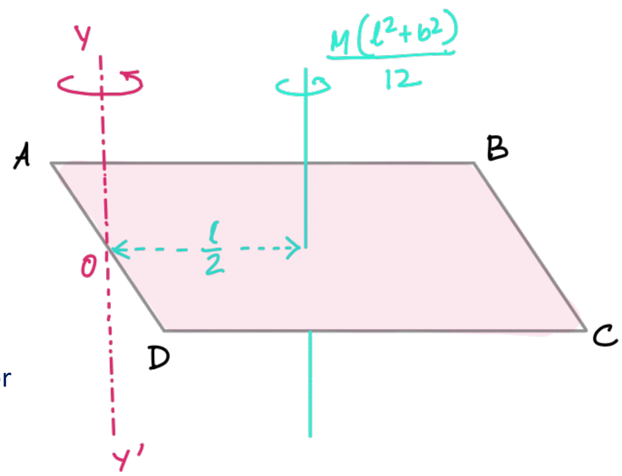
(iv) About an axis passing through the mid-point of one side and perpendicular to its plane

In this case the axis passes through the mid-point of side AD or BC, say, and perpendicular to the plane of the lamina, so that it is parallel to the axis through O (the c.m. of the lamina). This case can be obtained by an applying the principle of parallel axes to case (iii) above. Therefore, the M.I. of the lamina about this axis will be,

$$I = \frac{M(l^2+b^2)}{12} + M\left(\frac{l}{2}\right)^2$$

$$= M\left(\frac{l^2+b^2}{12} + \frac{l^2}{4}\right)$$

$$= M\left(\frac{l^2}{3} + \frac{b^2}{12}\right)$$



If the axis passes through the mid-point of AB or DC, we, similarly, get

$$I = \frac{M(l^2+b^2)}{12} + M\left(\frac{b}{2}\right)^2 = M\left(\frac{l^2+b^2}{12} + \frac{b^2}{4}\right)$$

$$= M\left(\frac{l^2}{12} + \frac{b^2}{3}\right)$$

(v) About an axis passing through one of its corners and perpendicular to its plane

Let the axis pass through the corner D of the lamina. Since it is perpendicular to the plane of the lamina, it is parallel to the axis through its centre of mass O in case (iii). Therefore, by the principle of parallel axes, we have moment of inertia of the rectangular lamina about this axis through D given by

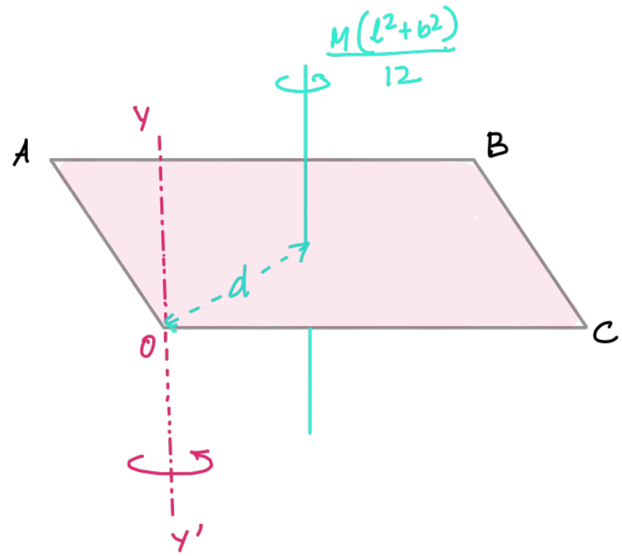
$I = \frac{M(l^2+b^2)}{12} + Md^2$, where d is the distance between the two axes, given by

$d^2 = (l/2)^2 + (b/2)^2 = (l^2 + b^2)/4$.

Therefore,

$$I = \frac{M(l^2+b^2)}{12} + \frac{M(l^2+b^2)}{4}$$

$$= M\left(\frac{l^2 + b^2 + 3l^2 + 3b^2}{12}\right) = \frac{M(l^2 + b^2)}{3}$$

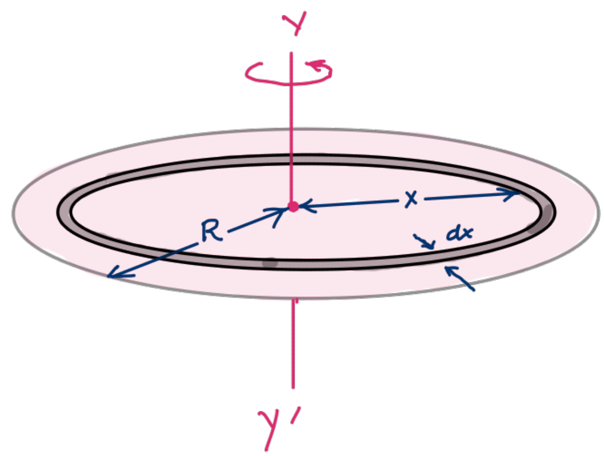


2. Moment of inertia of circular lamina or disc

(i) About an axis through its centre and perpendicular to its plane

Let M be the mass of the disc and R, its radius, so that its mass per unit area is equal to $M/\pi R^2$.

Considering a ring of the disc, of width dx at a distant x from the axis passing through O and perpendicular to the plane of the disc as shown in fig., we have



Area of the ring = Circumference × Width = $2\pi x.dx$

Mass of the ring
= $(M/\pi R^2) \times 2\pi x.dx = 2\,Mxdx/R^2$.

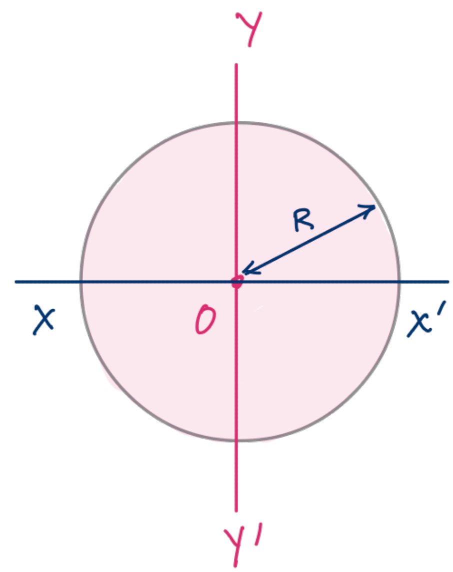
Therefore, its M.I. about the perpendicular axis through O = $\frac{2Mxdx}{R^2}x^2 = \frac{2Mx^3dx}{R^2}$.

The whole disc is supposed to be made up of such concentric rings of radii ranging from O to R, therefore, the M.I. of the whole disc about the axis through O and perpendicular to its plane, i.e., I, is obtained by integrating the above expression for the M.I. of the ring, between the limits x = 0 and x = R. Thus,

$$I = \int_0^R \frac{2M}{R^2}x^3dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2$$

(ii) About an axis along the diameter

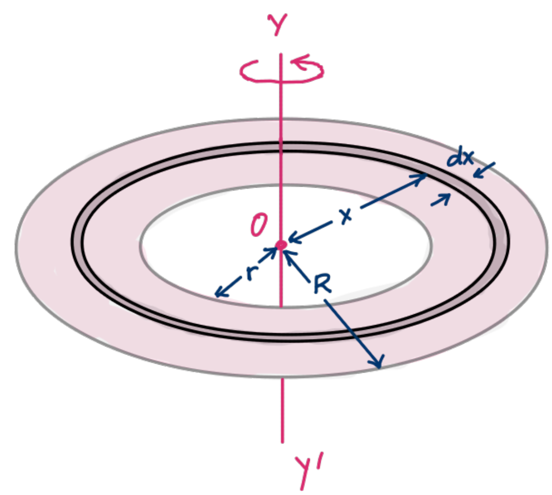
In this case, due to symmetry, the M.I. of the disc about one diameter is the same as about another. So, if I be the M.I. of the disc about each of the perpendicular diameters XOX′ and YOY′, (Fig.), we use the principle of perpendicular axes, i.e., I + I = M. I. of the disc about an axis through O and perpendicular to its plane, i.e., 2I = $MR^2/2$, which gives I = $MR^2/4$.



3. Moment of inertia of annular ring or annular disc

(i) About an axis through its centre and perpendicular to its plane

An annular disc is an ordinary disc, with a smaller coaxial disc removed from it, leaving a concentric circular hole in it, as shown in Fig.. If R and r are the outer and inner radii of the disc and M, its mass, we have mass per unit area of the disc = mass/area = $M/\pi(R^2-r^2)$.



The disc may be imagined to be made up of a number of circular rings, with their radii ranging from r to R. Considering one such ring of radius x and width dx, we have face area of the ring = 2πx.dx and, therefore, its mass = $2\pi x.dx.[M/\pi(R^2-r^2)] = [2Mx/(R^2-r^2)]\,dx$.

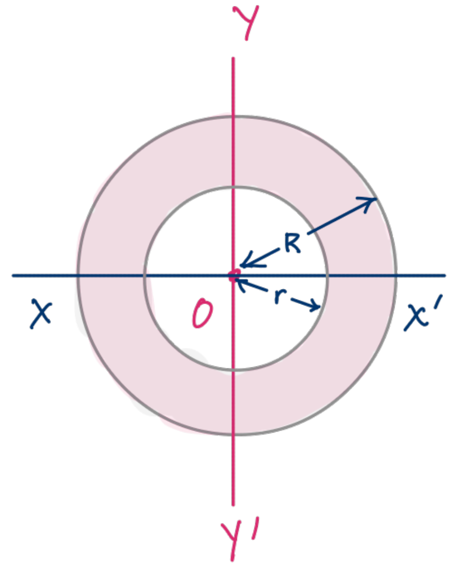
The M.I. about the axis through O and perpendicular to its plane $= \frac{2Mx}{(R^2-r^2)} dx.x^2 = \frac{2Mx^3}{(R^2-r^2)} dx$.

The M.I. of the whole annular disc i.e., I, is, therefore, given by the integral of the above expression between the limits x = r and x = R, i.e.,

$$I = \int_r^R \frac{2Mx^3}{(R^2-r^2)} dx = \frac{2M}{(R^2-r^2)} \int_r^R x^3 dx = \frac{2M}{(R^2-r^2)} \left[\frac{x^4}{4}\right]_r^R = \frac{2M}{(R^2-r^2)} \left[\frac{(R^4-r^4)}{4}\right] = \frac{M(R^2+r^2)}{2}$$

(ii) About an axis along the diameter

Due to symmetry, the M.I. of the annular disc about one diameter is the same as about another, say I. Then according to the principle of perpendicular axes, the sum of its moments of inertia about two perpendicular diameters must be equal to its M.I. about the axis passing through its centre (where the two diameters intersect) and perpendicular to its plane, i.e., $I + I = M(R^2+r^2)/2 \Longrightarrow 2I = M(R^2 + r^2)/2$, which gives, $I = M(R^2 + r^2)/4$.



4. Moment of inertia of a solid cylinder

(i) About its own axis of cylindrical symmetry

A solid cylinder is just a thick circular disc or a number of thin circular discs (all of the same radius) piled up one over the other, so that its axis of cylindrical symmetry is the same as the axis passing through the centre of the thick disc (or the pile of thin discs) and perpendicular to its plane.

Therefore, M.I. of the solid cylinder about its axis of cylindrical symmetry, i.e., I = M.I. of the thick disc (or the pile of thin discs) of the same mass and radius about the axis through its centre and perpendicular to its plane.

or, $I = MR^2/2$

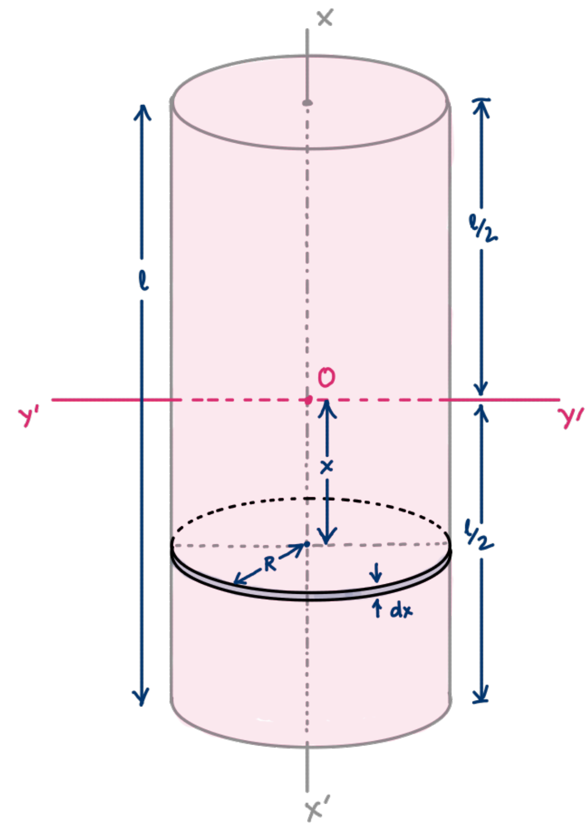
(ii) About the axis through its centre and perpendicular to its axis of cylindrical symmetry

If R be the radius, l be the length and M, the mass of the solid cylinder, supposed to be uniform and of homogeneous composition, we have its mass per unit length = M/l. Considering the cylinder to be made up of a number of discs each of radius R, placed adjacent to each other, and considering one such disc of thickness dx at a distance x from the centre O of the cylinder, (Fig.), we have mass of the disc = (M/l) dx, and radius = R.

Therefore, M.I. of the disc about its diameter (AB) $= \frac{M}{l}.dx.\frac{R^2}{4}$

The M.I. about the parallel axis YOY' , passing through the centre O of the cylinder and perpendicular to its axis of cylindrical symmetry (or its length), according to the principle of parallel axes,

$$= \frac{M}{l}.dx.\frac{R^2}{4} + \frac{M}{l}.dx.x^2$$



Therefore, the M.I. of the whole cylinder about this axis, i.e., I = twice the integral of the above expression between the limits x = 0 and x = l/2, i.e.,

$$I = 2\int_0^{\frac{l}{2}}\left(\frac{M}{l}.\frac{R^2}{4}dx + \frac{M}{l}x^2dx\right)$$

$$= \frac{2M}{l}\int_0^{\frac{l}{2}}\left(\frac{R^2}{4}dx + x^2dx\right)$$

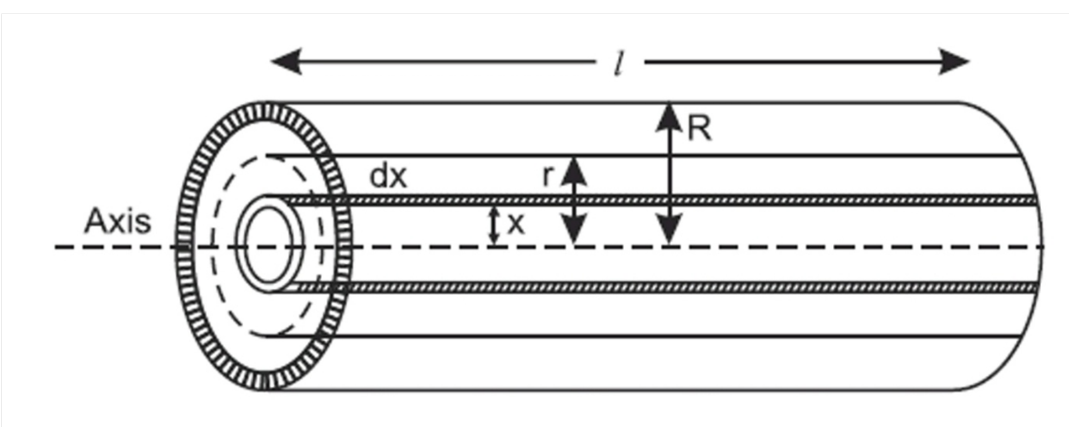
$$= \frac{2M}{l}\left[\frac{R^2x}{4} + \frac{x^3}{3}\right]_0^{\frac{l}{2}}$$

$$\Rightarrow I = \frac{2M}{l}\left[\frac{R^2}{4}.\frac{l}{2} + \frac{l^3}{8\times 3}\right] = \frac{2M}{l}\left[\frac{R^2l}{8} + \frac{l^3}{24}\right] = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)$$

5. Moment of inertia of a hollow cylinder

(i) About its axis of cylindrical symmetry

Let R and r be the external and internal radii respectively of a hollow cylinder of length l and mass M, (Fig.). Then, face-area of the cylinder = $\pi(R^2 - r^2)$ and its volume = $\pi(R^2 - r^2)l$ and, therefore, its mass per unit volume = $M/\pi(R^2 - r^2)l$. Considering the cylinder to be made up of a large number of thin, coaxial cylinders, with their radii varying from r to R, and considering one such cylinder of radius x and thickness dx, we have its face area = $2\pi x.dx$ and its volume = $2\pi xdx.l$ and hence its mass = $2\pi xdx.l \times M/\pi(R^2 - r^2)l$ = $2Mxdx/(R^2 - r^2)$.



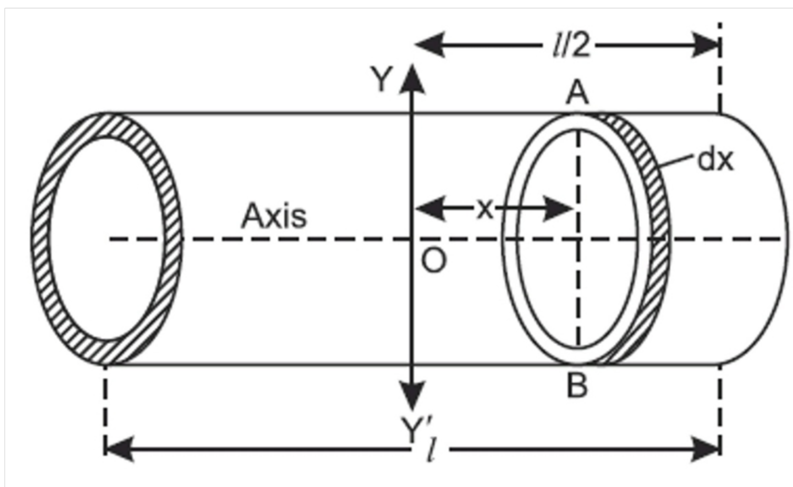
The M. I. about the axis of the cylinder $= \frac{2Mx}{(R^2-r^2)} dx . x^2 = \frac{2Mx^3}{(R^2-r^2)} dx$

Hence M. I. of the entire cylinder about its axis, i.e., I, is given by

$$I = \int_r^R \frac{2Mx^3}{(R^2-r^2)} dx = \frac{2M}{(R^2-r^2)} \int_r^R x^3 dx = \frac{2M}{(R^2-r^2)} \left[\frac{x^4}{4}\right]_r^R = \frac{2M}{(R^2-r^2)} \left[\frac{(R^4-r^4)}{4}\right] = \frac{M(R^2+r^2)}{2}$$

(ii) About an axis passing through its centre and perpendicular to its own axis

Again, if R and r be the external and internal radii respectively of the hollow cylinder, l, its length and M, its mass, we have mass per unit volume of the cylinder = $M/\pi(R^2 - r^2)l$. Considering the hollow cylinder to be made up of a large number of annular discs, of external and internal radii R and r respectively, placed adjacent to each other, and considering one such disc at a distance x from the axis YOY' passing through the centre O of the cylinder and perpendicular to its own axis as shown in Fig.. We have surface area of the disc = $\pi(R^2 - r^2)$, its volume = $\pi(R^2 - r^2)$ dx and, therefore, its mass = $\pi(R^2 - r^2)$ dx × $M/\pi(R^2 - r^2)l$ = Mdx/l.



The M. I. of the disc about its diameter (AB) $= \frac{M}{l} . dx . \frac{(R^2+r^2)}{4}$

Therefore, the M. I. about the parallel axis YOY' distant x from it, according to the principle of parallel axes, $= \frac{M}{l} . dx . \frac{(R^2+r^2)}{4} + \frac{M}{l} dx . x^2$

Therefore, M. I. of entire hollow cylinder about the axis YOY' is equal to twice the integral of the above expression between the limits x = 0 and x = l/2, i.e.,

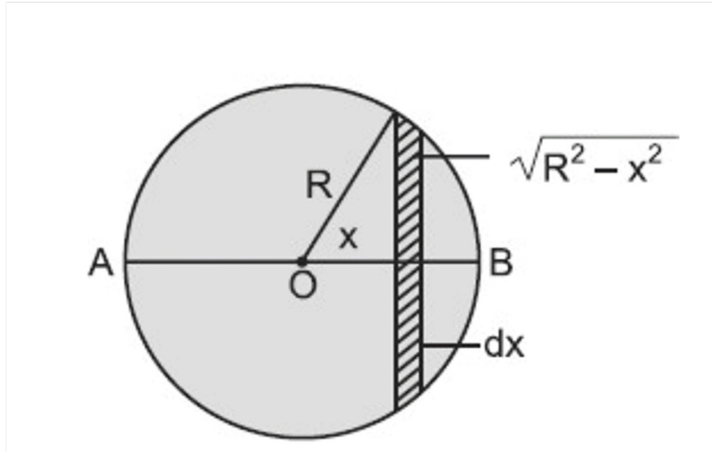
$$I = 2\int_0^{\frac{l}{2}} \left(\frac{M}{l} . \frac{(R^2+r^2)}{4} dx + \frac{M}{l} x^2 dx\right) = \frac{2M}{l} \int_0^{\frac{l}{2}} \left(\frac{(R^2+r^2)}{4} dx + x^2 dx\right) = \frac{2M}{l} \left[\frac{(R^2+r^2)x}{4} + \frac{x^3}{3}\right]_0^{\frac{l}{2}}$$

$$\Rightarrow I = \frac{2M}{l} \left[\frac{(R^2+r^2)}{4} . \frac{l}{2} + \frac{l^3}{8 \times 3}\right] = \frac{2M}{l} \left[\frac{(R^2+r^2)l}{8} + \frac{l^3}{24}\right] = M\left(\frac{(R^2+r^2)}{4} + \frac{l^2}{12}\right)$$

6. Moment of Inertia of a solid sphere

(i) About the diameter

In the fig. a section is illustrated, through the centre, of a solid sphere of radius R and mass M, whose moment of inertia is to be determined about a diameter AB, say, whose value is the same about any other diameter.



Since the volume of sphere = $4\pi R^3/3$, its mass per unit volume (or density) = $3M/4\pi R^3$.

Considering a thin circular slice of the sphere at a distance x from its centre O and of thickness dx, we have surface area of the slice (which is a disc of radius $\sqrt{R^2 - x^2}$) = $\pi(R^2 - x^2)$ and its Volume = Area × Thickness = $\pi(R^2 - x^2).dx$ and hence its mass = volume × density = $\pi(R^2 - x^2).dx \times 3M/4\pi R^3 = 3M(R^2 - x^2)\, dx/4R^3$.

The M.I. of this slice or disc about AB (i.e., an axis passing through its centre and perpendicular to its plane) = its mass × (radius)^2/2 $= \frac{3M(R^2-x^2)}{4R^3}dx.\frac{R^2-x^2}{2} = \frac{3M(R^2-x^2)^2}{8R^3}dx$

Therefore, M. I. of the whole sphere about its diameter (AB) is equal to twice the integral of the above expression between the limits x = 0 and x = R, i.e.,

$$I = 2\int_0^R \frac{3M(R^2-x^2)^2}{8R^3}dx = \frac{2\times 3M}{8R^3}\int_0^R (R^2-x^2)^2.dx = \frac{3}{4}\frac{M}{R^3}\int_0^R (R^4 - 2R^2x^2 + x^4).dx$$

$$\Rightarrow I = \frac{3M}{4R^3}\left[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^R = \frac{3M}{4R^3}\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right) = \frac{3M}{4R^3}\times\frac{8R^5}{15} = \frac{2}{5}MR^2$$

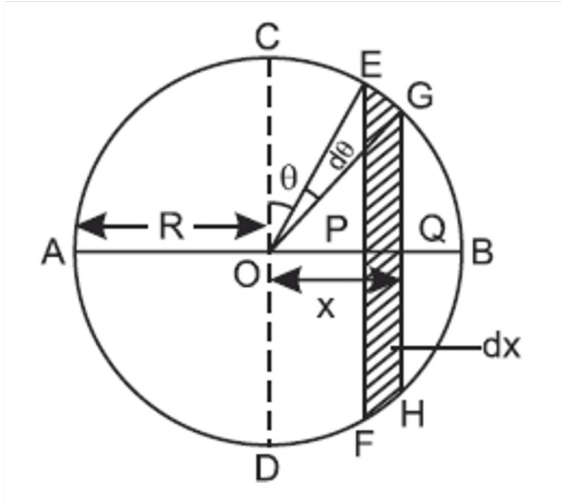
(ii) About a tangent

A tangent drawn to the sphere at any point will clearly be parallel to any diameter of it (i.e., an axis passing through its centre or centre or mass) and at a distance equal to the radius of the sphere, R, from it. We therefore have, by the principle of parallel axes, moment of inertia of the sphere about a tangent,

$$I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

7. Moment of inertia of a spherical shell

(i) About its diameter



Let ABCD be the section through the centre O, of a spherical shell of radius R and mass M, (Fig.) whose moment of inertia is to be determined about a diameter AB, say, its value being obviously the same about any other diameter. Then the surface area of the shell = $4\pi r^2$ and, therefore, mass per unit area of the shell = $M/4\pi R^2$. Considering a thin slice of the shell, lying between two planes EF and GH, perpendicular to the diameter AB at distances x and x + dx respectively from its centre O. This slice is a ring of radius PE and width EG (and not PQ which is equal to dx, the distance between the two planes). Then the area of the ring = circumference × width = 2πPE × EG and hence its mass = $2\pi PE \times EG \times M/4\pi R^2$.

Let us join OE and OG and let ∠COE be equal to θ and ∠EOG = dθ. Then, PE = OE cos ∠OEP = R cos θ, because OE = R and ∠OEP = alternate ∠COE = θ. Also OP = OE sin ∠OEP ⟹ x = R sin θ

and ∴ dx/dθ = R cos θ. [as OP = x and OE = R].

⟹ dx = R cos θ d θ = PE. dθ and EG = OE.dθ = R.dθ.

∴ mass of the ring = 2π PE × R dθ × $M/4\pi R^2$ = Mdx/2R. [as PE.dθ = dx]

The M. I. of the ring about diameter AB of the shell (i.e., an axis passing through the centre of the ring and perpendicular to its plane = mass × $(\text{radius})^2$ = (Mdx/2R) $(R^2 - x^2)$, [because $PE^2 = OE^2 - OP^2 = R^2 - x^2$]

Therefore, M.I. of the whole spherical shell about the diameter (AB) is equal to twice the integral of this expression between the limits x = 0 and x = R, i.e.,

$$I = 2\int_0^R \frac{M}{2R}(R^2 - x^2)dx = \frac{M}{R}\int_0^R (R^2 - x^2)dx = \frac{M}{R}\left[R^2x - \frac{x^3}{3}\right]_0^R = \frac{M}{R}\left[R^3 - \frac{R^3}{3}\right] = \frac{M}{R}.\frac{2}{3}R^3 = \frac{2}{3}MR^2$$

(ii) About a tangent

A tangent drawn to the sphere at any point will clearly be parallel to any diameter of it (i.e., an axis passing through its centre or centre or mass) and at a distance equal to the radius of the sphere, R, from it. By the principle of parallel axes, M.I. of the sphere about a tangent,

$$I = \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2$$

8. Moment of inertia of a hollow sphere or a thick shell

(i) About its diameter

A hollow sphere (or a thick shell) is just a solid sphere from the inside of which a small concentric solid sphere has been removed. The M.I. of hollow sphere about a diameter = M.I. of the solid sphere minus M.I. of the smaller solid sphere removed from it, both about the same diameter. Let R and r be the external and internal radii of the hollow sphere, i.e., the radius of the bigger solid sphere and the smaller solid sphere (removed from it) respectively. If ρ be the density of the material of the given hollow sphere, then

Mass of the bigger sphere $= \frac{4}{3}\pi R^3 \rho$ and

Mass of the smaller sphere $= \frac{4}{3}\pi r^3 \rho$

Thus, mass of the hollow sphere, $M = \frac{4}{3}\pi(R^3 - r^3)\rho$, and therefore, $\rho = \frac{3M}{4\pi(R^3 - r^3)}$

M.I. of the bigger and the smaller spheres about a given diameter are respectively $\frac{2}{5}\left(\frac{4}{3}\pi R^3 \rho\right) R^2$ and $\frac{2}{5}\left(\frac{4}{3}\pi r^3 \rho\right) r^2$.

Therefore. M.I. of the hollow sphere about the same diameter is given by

$$I = \frac{2}{5}\left(\frac{4}{3}\pi R^3 \rho\right) R^2 - \frac{2}{5}\left(\frac{4}{3}\pi r^3 \rho\right) r^2 = \frac{2}{5}.\frac{4}{3}\pi\rho(R^5 - r^5) = \frac{8}{15}\pi \frac{3M}{4\pi(R^3 - r^3)}(R^5 - r^5) = \frac{2}{5}M\left(\frac{R^5 - r^5}{R^3 - r^3}\right)$$

(ii) About a tangent

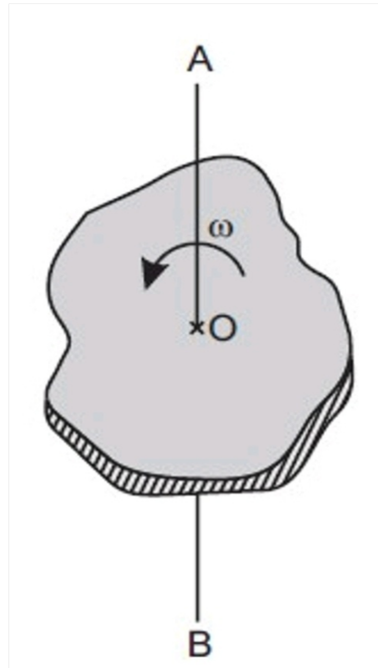
A tangent to the sphere at any point is parallel to any diameter (i.e., the axis passing through the centre of mass) of the sphere and at a distance equal to its external radius R from it, then according to the principle of parallel axes,

M. I. of hollow sphere about a tangent, $I = \frac{2}{5}M\left(\frac{R^5 - r^5}{R^3 - r^3}\right) + MR^2$

Kinetic energy of rotation

Speaking about kinetic energy of rotation, there are two cases:

1. Kinetic energy of a body rotating about an axis through its centre of mass (i.e., in the case of pure rotation).



Let us consider a body of mass M, rotating with angular velocity ω about an axis AB, passing through its centre of mass, O (Fig.), so that the centre of mass has zero linear velocity. It is thus a case of pure rotation.

Then the body possesses kinetic energy in virtue of its motion of rotation which is, therefore, aptly called its kinetic energy of rotation. Let us obtain an expression for it. Let us consider the rigid body having large number of particles of masses m_1, m_2, m_3...etc. at respective distances r_1, r_2, r_3...etc. from the axis AB through O. Since their angular velocity is the same (ω), their linear velocities are respectively $r_1\omega = v_1$, $r_2\omega = v_2$, $r_3\omega = v_3$, ... etc. and hence their respective kinetic energies equal to $\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2$; $\frac{1}{2}m_2v_2^2 = \frac{1}{2}m_2r_2^2\omega^2$; $\frac{1}{2}m_3v_3^2 = \frac{1}{2}m_3r_3^2\omega^2$; ... etc.

Therefore total K.E. of all the particles, i.e., the K.E. of the rigid body itself

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \cdots$$

$$= \frac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ \ldots) = \frac{1}{2}\omega^2\sum mr^2 = \frac{1}{2}\omega^2MK^2,$$

where $\sum mr^2 = MK^2$, with M, as the mass of the body and K, its radius of gyration about the axis of rotation AB. Since $MK^2 = I$, the moment of inertia of the body about the axis AB.

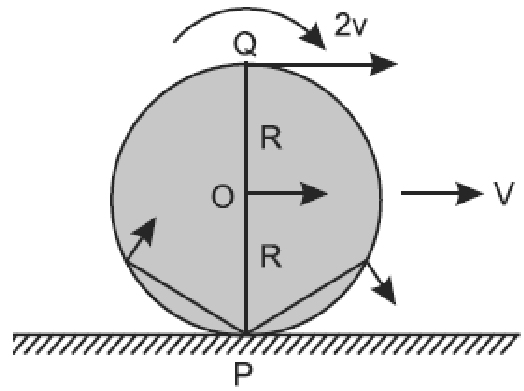
Then the Kinetic Energy of rotation of the body about the axis AB through its centre of mass = $\frac{1}{2}I\omega^2$.

2. Kinetic energy of a rotating body whose centre of mass also has a linear velocity

(a) Case of a body rolling along a plane surface

Let us consider a body, like a circular disc, a cylinder, a sphere etc. (i.e., a body with a circular symmetry), of mass M, radius R and with its centre of mass at O, (Fig.), rolling, without slipping, along a plane or a level surface, such that it rotates clockwise and moves along the + x direction, as indicated.

At any given instant, the point P, where the body touches the surface, is at rest, so that an axis through P, perpendicular to the plane of the paper is its instantaneous axis of rotation and the linear velocities of its various particles are perpendicular to the lines joining them with the point of contact P, their magnitudes being proportional to the lengths of these lines, as shown by the directions and lengths of the arrows at the various points. Thus, if the linear velocity of the centre of mass O (where PO = R) be v, that of the particle at Q (where PQ = 2R) is 2v.



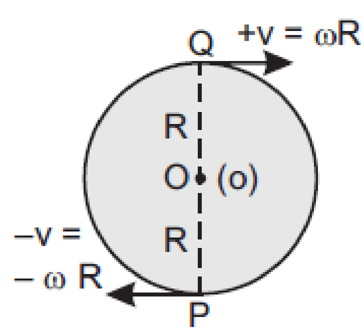
This means clearly that the particles have all the same angular velocity with respect to the point P or that the body is rotating about the fixed axis through P with an angular velocity ω, say, given by v/R, where v is the linear velocity of the centre of mass. The motion of the body is thus equivalent to one of pure rotation about the axis through P, with an angular velocity ω. The whole of the kinetic energy of the body is, therefore, the same as its kinetic energy of rotation about this axis and hence equal to $(1/2)I_P\omega^2$, where I_P is the M. I. of the body about the axis through R. If $I_{c.m.}$ is the M.I. of the body about a parallel axis through its centre of mass, then according to the principle of parallel axes, $I_P = I_{c.m.} + MR^2$.

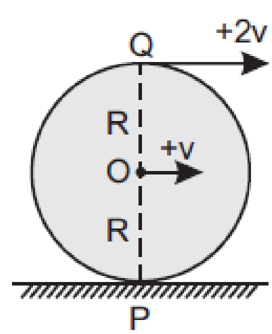
The K.E. of the rolling body = $\frac{1}{2}(I_{c.m.} + MR^2)\omega^2 = \frac{1}{2}I_{c.m.}\omega^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}I_{c.m.}\omega^2 + \frac{1}{2}Mv^2$ (1)

where v is the linear speed of its centre of mass with respect to P. Here $I_{c.m.} = MK^2$, where K is the radius of gyration of the body about the axis through its centre of mass, and ω = v/R.

Thus K.E. of the rolling body $= \frac{1}{2}MK^2\frac{v^2}{R^2} + \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2\left(\frac{K^2}{R^2} + 1\right)$ (2)

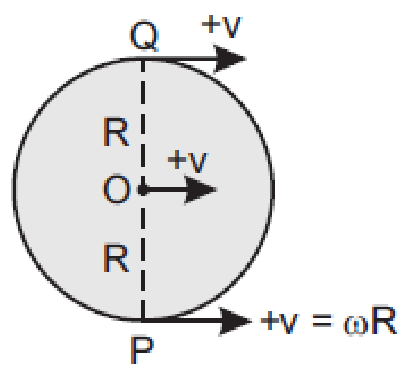
In the above expression, one can see two terms on R.H.S, first one gives its K.E. of pure rotation about the centre of mass, i.e., its K.E. when it is simply rotating with angular velocity ω about the axis through its centre of mass, without executing any translational motion (i.e., with the linear velocity of its centre of mass being zero). The second term gives its K. E. of pure translation, i.e., its K.E. when it is simply moving with linear velocity v (or the linear speed of the centre of mass) without performing any rotational motion (i.e., with





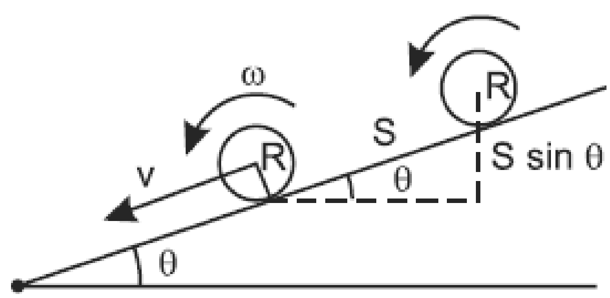
its angular velocity being zero).

Therefore, K. E. of a rolling body rotating with angular velocity ω and moving with linear velocity v (= Rω) = its K. E. of pure rotation (with the same angular velocity ω) about its centre of mass + its K. E. of pure translation, with its centre of mass moving with linear velocity v.



This can be applied to all bodies, rolling or otherwise, which are simultaneously executing a translational motion and a rotational motion about an axis perpendicular to their planes of motion. Three figures beside reveal interesting aspects. Fig. represents the rotation of the body about the fixed axis through the point of contact P, where the linear velocity of the particle at P is zero, that of the centre of mass, + v along the + x direction and that of the particle at Q, + 2v in the + x direction. Fig. represents the pure rotation of the body about the axis through its centre of mass O, when it is at rest, i.e., its linear velocity is zero, that of the particle at P, - v = -ωR along the -x direction and that of the particle at Q, + v = ωR along the + x direction. Fig. represents the pure translational motion of the body, with the linear velocity of the centre of mass O equal to + v so that the linear velocities of all other particles at P, Q etc are also the same, i.e., + v.

(b) Case of a body rolling down an inclined plane - Its acceleration along the plane



Let a body of circular symmetry (e.g., a disc, sphere, cylinder etc.,) of mass M, roll freely down a plane inclined to the horizontal at an angle θ, (Fig.) and rough enough to prevent slipping. If v be the linear velocity acquired by the body on covering a distance S along the plane, its vertical distance of descent = S sin θ.

Then the potential energy lost by the body = Mg. S sin θ. This must be equal to the kinetic energy gained by the body, i.e., equal to its K. E. of rotation plus its K. E. of translation.

The K. E. of rotation of the body = $\frac{1}{2}I\omega^2$, where ω is its angular velocity about a perpendicular axis through its centre of mass, and its K. E. of translation = $\frac{1}{2}Mv^2$, because its centre of mass has a linear velocity v.

The total K.E. gained by the body = $\frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$. Here I = MK^2, where K is the radius of gyration of the body about the axis through its centre of mass, and ω = v/R.

Total K.E. gained by the body $= \frac{1}{2}MK^2\frac{v^2}{R^2} + \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2\left(\frac{K^2}{R^2} + 1\right)$

Equating this gain of K. E. against the loss of P. E., we have

$$\frac{1}{2}Mv^2\left(\frac{K^2}{R^2}+1\right) = Mg\,S\sin\theta \Longrightarrow v^2 = 2\frac{R^2}{K^2+R^2}g.S.\sin\theta$$

Comparing this with the kinematic relation $v^2 = 2aS$ for a body starting from rest, we have acceleration of the body along the plane,

i.e., $a = \frac{R^2}{K^2+R^2}g.\sin\theta$

i.e., the acceleration is proportional to $R_2/(K_2 + R_2)$ for a given angle of inclination (θ) of the plane.

Here it is clear that the greater the value of K, as compared with R, the smaller the acceleration of the body rolling down along the plane and hence the greater the time taken by it in reaching the bottom of the plane, and the acceleration, and hence the time of descent, is quite independent of the mass of the body.

Special cases:

(i) Cylinder.

The moment of inertia of a cylinder about its axis of symmetry about which it rolls = $\frac{1}{2}MR^2 = MK^2$

$\Longrightarrow \frac{K^2}{R^2} = \frac{1}{2}$. Hence, its acceleration, $a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}} = \frac{g\sin\theta}{1+\frac{1}{2}} = \frac{2}{3}\ g\sin\theta$

(ii) Solid Sphere

The moment of inertia of a solid sphere about a diameter about which it rolls = $\frac{2}{5}MR^2 = MK^2$

$\Longrightarrow \frac{K^2}{R^2} = \frac{2}{5}$. Hence, its acceleration, $a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}} = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5}{7}\ g\sin\theta$

(iii) Hollow sphere

The moment of inertia of a hollow sphere about a diameter about which it rolls = $\frac{2}{3}MR^2 = MK^2$

$\Longrightarrow \frac{K^2}{R^2} = \frac{2}{3}$. Hence, its acceleration, $a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}} = \frac{g\sin\theta}{1+\frac{2}{3}} = \frac{3}{5}\ g\sin\theta$