Solved Problems – Unit I (Vector Calculus)
From Electrodynamics by D.J. Griffiths
Himanshu Bora

Example 1.1 **Example 1.1.** Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$ (Fig. 1.7), and calculate the dot product of $\mathbf{C}$ with itself.

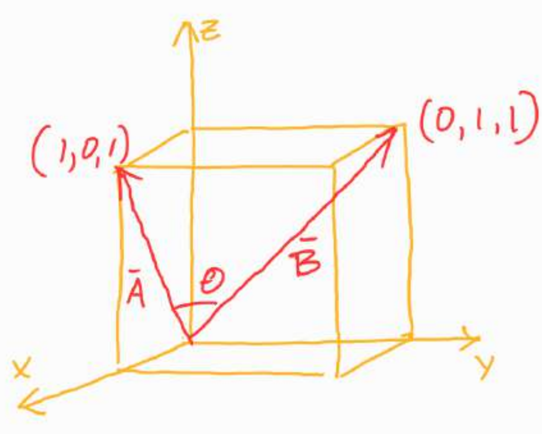
Solution :

$$\bar{C} = \bar{A} - \bar{B}$$

$$\begin{aligned}
\bar{C}\cdot\bar{C} &= (\bar{A}-\bar{B})\cdot(\bar{A}-\bar{B}) \\
&= \bar{A}\cdot\bar{A} - \bar{B}\cdot\bar{A} - \bar{A}\cdot\bar{B} + \bar{B}\cdot\bar{B} \\
&= A^2 + B^2 - 2\,\bar{A}\cdot\bar{B} \quad \text{(since dot product is commutative)} \\
&= A^2 + B^2 - 2AB\cos\theta
\end{aligned}$$

Example 1.2 **Example 1.2.** Find the angle between the face diagonals of a cube.

Solution :-



$$\bar{A} = \hat{x} + \hat{z} \;;\quad A = \sqrt{2}$$
$$\bar{B} = \hat{y} + \hat{z} \;;\quad B = \sqrt{2}$$

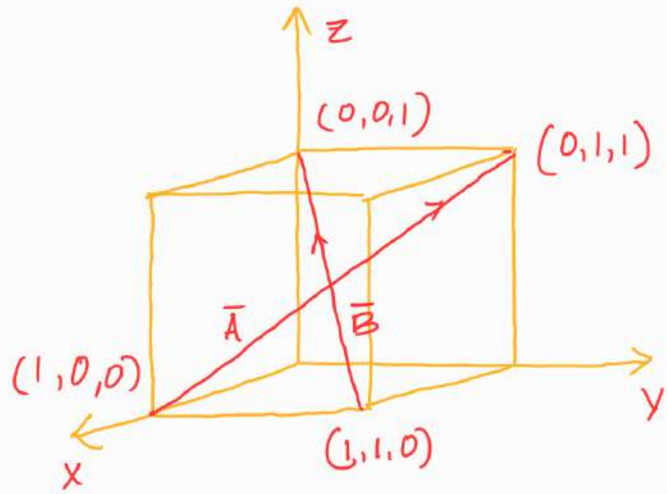
$$\bar{A}\cdot\bar{B} = AB\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\bar{A}\cdot\bar{B}}{AB} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$ //

Problem 1.3 **Problem 1.3** Find the angle between the body diagonals of a cube.

Solution :-



$$\bar{A} = -\hat{x} + \hat{y} + \hat{z} \quad ; \quad A = \sqrt{3}$$
$$\bar{B} = -\hat{x} - \hat{y} + \hat{z} \quad ; \quad B = \sqrt{3}$$

$$\bar{A} \cdot \bar{B} = AB\cos\theta$$
$$\Rightarrow \cos\theta = \frac{\bar{A} \cdot \bar{B}}{AB}$$
$$= \frac{1 - 1 + 1}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \qquad \therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Problem 1.4 :- **Problem 1.4** Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the shaded plane in Fig. 1.11.

Solution

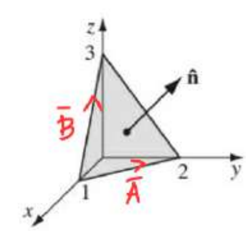


FIGURE 1.11

$$\bar{A} = -1\hat{x} + 2\hat{y} \quad ; \quad A = \sqrt{5}$$
$$\bar{B} = -1\hat{x} + 3\hat{z} \quad ; \quad \bar{B} = \sqrt{10}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \hat{x}(6-0) - \hat{y}(-3-0) + \hat{z}(0+2) = 6\hat{x} + 3\hat{y} + 2\hat{z}$$

$$\hat{n} = \frac{\bar{A} \times \bar{B}}{|\bar{A} \times \bar{B}|} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{\sqrt{6^2 + 3^2 + 2^2}} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{\sqrt{49}}$$

$$\hat{n} = \frac{6}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{2}{7}\hat{z}$$

Problem 1.7

Problem 1.7 Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point (2,8,7) to the field point (4,6,8). Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.

Solution :-

** seperation → field − source

$$\bar{\mathscr{r}} = (r - r')$$
$$= (4-2)\hat{x} + (6-8)\hat{y} + (8-7)\hat{z}$$
$$= 2\hat{x} - 2\hat{y} + \hat{z}$$

magnitude $\mathscr{r} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$

$$\hat{\mathscr{r}} = \frac{\bar{\mathscr{r}}}{\mathscr{r}} = \frac{2}{3}\hat{x} - \frac{2}{3}\hat{y} + \frac{1}{3}\hat{z}$$

//

Example 1.3

Example 1.3. Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

Solution

$$\nabla r = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= \hat{x}\frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x + \hat{y}\frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y$$

$$+ \hat{z}\frac{1}{2\sqrt{x^2+y^2+z^2}} 2z$$

$$= \frac{x\hat{x} + y\hat{y} + z\hat{x}}{\sqrt{x^2+y^2+z^2}} = \frac{\bar{r}}{r} = \hat{r}$$

//

Problem 1.11

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Solution :-

a) $f = x^2 + y^3 + z^4$

$$\nabla f = 2x\,\hat{x} + 3y^2\,\hat{y} + 4z^3\,\hat{z}$$

//

b) $f = x^2 y^3 z^4$

$$\nabla f = 2x\,y^3 z^4\,\hat{x} + x^2\,3y^2 z^4\,\hat{y} + x^2 y^3\,4z^3\,\hat{z}$$

//

c) $f = e^x \sin(y) \ln(z)$

$$\nabla f = e^x \sin(y) \ln(z)\,\hat{x} + e^x \cos(y) \ln(z)\,\hat{y} + e^x \sin(y)\,\frac{1}{z}\,\hat{z}$$

//

Problem 1.12

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

Solution :

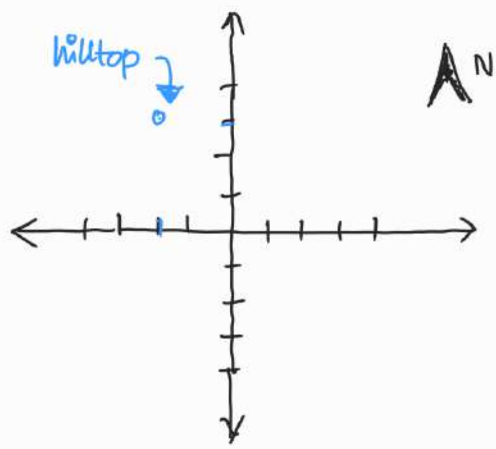
a) Calculating top of the hill

For calculating maxima, setting $\bar{\nabla} h = \bar{0}$.

$$\Rightarrow 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y} = \bar{0}$$

$$2y - 6x - 18 = 0 \quad \text{and} \quad 2x - 8y + 28 = 0 \rightarrow (ii)$$

$$\Rightarrow 8y - 24x - 72 = 0 \rightarrow (i)$$



$$(i) + (ii) \Rightarrow -22x - 44 = 0$$

$$\Rightarrow x = -\frac{44}{22} = -2$$

From (ii)

$$8y = 2x + 28$$

$$8y = 2(-2) + 28$$

$$\Rightarrow y = \frac{1}{8}(24) = 3$$

Thus, the top of the hill is located 3 miles north and 2 miles east of South Hadley.

b) Calculation of hill height $\quad\quad \begin{array}{c} 28 \\ 3 \\ \hline 24 \end{array}\,2$

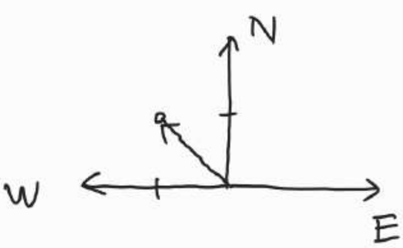
$$h(x,y)\Big|_{(-2,3)} = 10\big(2(-2)\cdot 3 - 3\cdot(-2)^2 - 4\cdot 3^2 - 18(-2) + 28\cdot 3 + 12\big)$$

$$= 10\left(-12 - \cancel{12} - \cancel{36} + \cancel{36} + 84 + \cancel{12}\right)$$
$$= 10\,(72)$$
$$= 720 \text{ ft}$$

//

c) $\nabla h = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$

$$\bar{\nabla} h\Big|_{(1,1)} = 10(2 - 6 - 18)\hat{x} + 10(2 - 8 + 28)\hat{y}$$
$$= 10(-22)\hat{x} + 10(22)\hat{y}$$
$$= -220\hat{x} + 220\hat{y}$$



$$|\bar{\nabla} h| = \sqrt{(220)^2 + (220)^2}$$
$$= 220\sqrt{2}$$

Thus, the hill is $220\sqrt{2}$ miles/ft steep towards the north west direction

At that point the hill is steepest towards the north-west direction

Problem 1.13

Problem 1.13 Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

Solution :-

a) $$\nabla(\mathscr{r}^2) = \nabla\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]$$

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]$$

$$= 2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}$$

$$= 2\vec{\mathscr{r}}$$

b) $$\nabla\left(\frac{1}{\mathscr{r}}\right) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right)$$

$$= -\frac{1}{2}\frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}\left[2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}\right]$$

$$= -\frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}$$

e) $\nabla(r^n)$

In sph. polar co-ordinates.

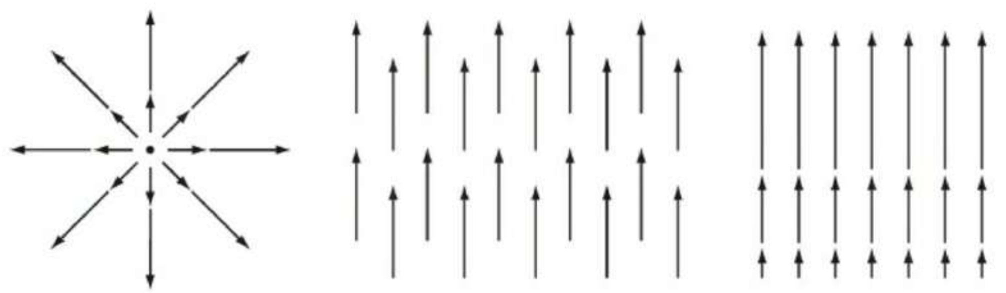
$$\bar{\nabla} \equiv \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\therefore \bar{\nabla}(r^n) = \hat{r}\frac{\partial}{\partial r}(r^n)$$

$$= n r^{n-1}\hat{r}$$

Example 1.4

Example 1.4. Suppose the functions in Fig. 1.18 are $\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, and $\mathbf{v}_c = z\,\hat{\mathbf{z}}$. Calculate their divergences.



(a) (b) (c)

FIGURE 1.18

Solution :-

$$\bar{\nabla}\cdot\bar{v}_a = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1+1+1 = 3$$

$$\bar{\nabla}\cdot\bar{v}_b = \frac{\partial}{\partial z}(1) = 0$$

$$\bar{\nabla}\cdot\bar{v}_c = \frac{\partial}{\partial z}(z) = 1$$

Problem 1.15

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

a). $$\bar{\nabla}\cdot\bar{v}_a = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(xz^2) + \frac{\partial}{\partial y}(-2xz)$$

$$= 2x + 0 - 2x = 0$$

b) $$\bar{\nabla}\cdot\bar{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx)$$

$$= y + 2z + 3x$$

c) $$\bar{\nabla}\cdot\bar{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= 0 + 2x + 2y = 2x + 2y$$

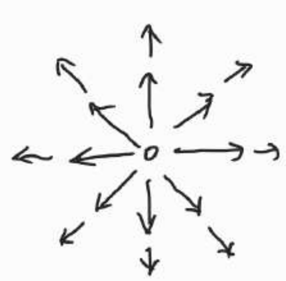
Problem 1.16 -

Problem 1.16 Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you... can you explain it?

Solution :



$$v = \frac{1}{r^2}\hat{r}$$

Div. using the usual formula.

$$\bar{\nabla}\cdot\bar{v} = \frac{\partial}{\partial r}\cdot\frac{1}{r^2} = \frac{-2}{r^3}$$

Calculation using the usual divergence formula gives a negative divergence. But from the sketch we can see that the divergence is positive.

The reason we cannot compute divergence for this is — the function $\frac{1}{r^2}$ blows up at $x = 0$. //

Problem 1.18

Problem 1.18 Calculate the curls of the vector functions in Prob. 1.15.

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

Solution :-

a) $$\bar{\nabla} \times \bar{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$$

$$= \hat{x}\left\{\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2)\right\}$$

$$-\hat{y}\left\{\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial z}x^2\right\} +$$

$$\hat{z}\left\{\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}x^2\right\}$$

$$= -6xz\,\hat{x} + 2z\,\hat{y} + 3z^2\,\hat{z}$$ //

b) $$\bar{\nabla} \times \bar{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= \hat{x}\left\{\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz)\right\} - \hat{y}\left\{\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial z}(xy)\right\}$$

$$+ \hat{z}\left\{\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial y}(xy)\right\}$$

$$= \hat{x}\{0 - 2y\} - \hat{y}\{3z - 0\} + \hat{z}\{0 - x\}$$

$$= -2y\hat{x} - 3z\hat{y} - x\hat{z}$$

c) $$\bar{\nabla} \times \bar{v}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix}$$

$$= \hat{x}\left\{\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy + z^2)\right\} - \hat{y}\left\{\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z} y^2\right\}$$

$$+ \hat{z}\left\{\frac{\partial}{\partial x}(2xy + z^2) - \frac{\partial}{\partial y} y^2\right\}$$

$$= \hat{x}\{2z - 2z\} - \hat{y}\{2y - 2y\} + \hat{z}\{2y - 2y\}$$

$$= 0$$

Problem 1.22

(a) If **A** and **B** are two vector functions, what does the expression $(\mathbf{A}\cdot\nabla)\mathbf{B}$ mean? (That is, what are its x, y, and z components, in terms of the Cartesian components of **A**, **B**, and ∇?)

(b) Compute $(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector defined in Eq. 1.21.

(c) For the functions in Prob. 1.15, evaluate $(\mathbf{v}_a\cdot\nabla)\mathbf{v}_b$.

Solution :-

a) $(\bar{A}\cdot\bar{\nabla})\bar{B}$

$$= \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}\right)\left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right)$$

$$= A_x \frac{\partial B_x}{\partial x}\hat{x} + A_x B_x \underbrace{\frac{\partial}{\partial x}\hat{x}}_{=0} + A_x \frac{\partial B_y}{\partial x}\hat{y} + A_x B_y \underbrace{\frac{\partial}{\partial x}\hat{y}}_{=0}$$

$$+ A_x \frac{\partial B_z}{\partial x}\hat{z} + A_y B_z \underbrace{\frac{\partial \hat{z}}{\partial x}}_{=0} + -----$$

$$= A_x \frac{\partial B_x}{\partial x}\hat{x} + A_x \frac{\partial B_y}{\partial x}\hat{y} + A_x \frac{\partial B_z}{\partial x}\hat{z} +$$

$$A_y \frac{\partial B_x}{\partial y}\hat{x} + A_y \frac{\partial B_y}{\partial y}\hat{y} + A_z \frac{\partial B_z}{\partial y}\hat{z}$$

$$A_z \frac{\partial B_x}{\partial z}\hat{x} + A_z \frac{\partial B_y}{\partial z}\hat{y} + A_z \frac{\partial B_z}{\partial z}\hat{z}$$

b) $(\hat{r}\cdot\nabla)\hat{r}$

$$= \left\{ \hat{r} \cdot \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial}{\partial \phi} \right) \right\} \hat{r}$$

$$= \frac{\partial}{\partial r}(\hat{r})$$

$$= 0$$ unit vector donot change w.r.t 'r'

//

c)

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

$$\bar{v}_a \cdot \bar{\nabla} = x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}$$

$$(\bar{v}_a\cdot\bar{\nabla})\, v_b = x^2(y\hat{x} + 3z\hat{z}) + 3xz^2(x\hat{x} + 2z\hat{y}) + -2xz(2y\hat{y} + 3x\hat{x})$$

$$= (x^2y + 3x^2z^2)\,\hat{x} + (6xz^3 - 4xyz)\,\hat{y} - 3x^2z\,\hat{z}$$

Problem 1.25

Problem 1.25

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) Do the same for rule (vi).

Solution :-

a) Product rule iv)

$$\bar{\nabla}\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\bar{\nabla}\times\bar{A}) - \bar{A}\cdot(\bar{\nabla}\times\bar{B})$$

$$\text{LHS} = \bar{\nabla}\cdot(\bar{A}\times\bar{B})$$

$$= \bar{\nabla}\cdot\left\{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix}\right\}$$

$$= \bar{\nabla}\cdot\left\{\hat{x}(6xz) - \hat{y}(0 - 9yz) + \hat{z}(-2x^2 - 6y^2)\right\}$$

$$= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}(-2x^2 - 6y^2)$$

$$= 6z + 9z = 15z$$

$$\text{RHS} = \bar{B}\cdot(\bar{\nabla}\times\bar{A}) - \bar{A}\cdot(\bar{\nabla}\times\bar{B})$$

$$= \bar{B} \cdot \left\{ \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} \right\} - \bar{A} \cdot \left\{ \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \right\}$$

$$= \left(3y\hat{x} - 2x\hat{y}\right) \left\{ \hat{x}(0) - \hat{y}(0) + \hat{z}(0) \right\}$$

$$- \left(x\hat{x} + 2y\hat{y} + 3z\hat{z}\right) \left\{ \hat{x}(0) - \hat{y}(0) + \hat{z}(-2-3) \right\}$$

$$= -3z(-5) = 15z$$

LHS = RHS

b) Product Rule (ii)

$$\bar{\nabla}(\bar{A} \cdot \bar{B}) = \bar{A} \times (\bar{\nabla} \times \bar{B}) + \bar{B} \times (\bar{\nabla} \times \bar{A}) + (\bar{A} \cdot \bar{\nabla}) B + (\bar{B} \cdot \bar{\nabla}) \bar{A}$$

$$\text{LHS} = \bar{\nabla}(\bar{A} \cdot \bar{B}) = \bar{\nabla}\left[\left(x\hat{x} + 2y\hat{y} + 3z\hat{z}\right)\left(3y\hat{x} - 2x\hat{y}\right)\right]$$

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(3xy - 4xy\right)$$

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(-xy\right)$$

$$= -y\hat{x} - x\hat{y}$$

$$\bar{\nabla}\times\bar{A} = 0 \quad ; \quad \bar{\nabla}\times\bar{B} = -5\hat{z}$$

$$(\bar{A}.\bar{\nabla})\bar{B} = \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)(3y\hat{x} - 2x\hat{y})$$

$$= x(-2\hat{y}) + 2y(3\hat{x})$$

$$= 6y\hat{x} - 2x\hat{y}$$

$$(\bar{B}.\bar{\nabla})\bar{A} = \left(3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}\right)(x\hat{x} + 2y\hat{y} + 3z\hat{z})$$

$$= 3y(\hat{x}) - 2x(2\hat{y})$$

$$= 3y\hat{x} - 4x\hat{y}$$

$$RHS = \bar{A}\times(\bar{\nabla}\times\bar{B}) + \bar{B}\times(\bar{\nabla}\times\bar{A}) + (\bar{A}.\bar{\nabla})B + (\bar{B}.\bar{\nabla})\bar{A}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} + 0 + 6y\hat{x} - 2x\hat{y} + 3y\hat{x} - 4x\hat{y}$$

$$= \hat{x}(-10y) - \hat{y}(-5x) + 6y\hat{x} - 2x\hat{y} + 3y\hat{x} - 4x\hat{y}$$

$$= (-10y + 6y + 3y)\hat{x} + (+5x - 2x - 4x)\hat{y}$$

$$= -y\hat{x} - x\hat{y}$$

LHS = RHS.

c) Product Rule (vi)

$$\bar{\nabla}\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\bar{\nabla})\bar{A} - (\bar{A}\cdot\bar{\nabla})\bar{B} + \bar{A}(\bar{\nabla}\circ\bar{B}) - \bar{B}(\bar{\nabla}\circ\bar{A})$$

$$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} = \hat{x}(6xz) - \hat{y}(-9yz) + \hat{z}(-2x^2 - 6y^2)$$

$$LHS = \bar{\nabla}\times(\bar{A}\times\bar{B})$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & (-2x^2 - 6y^2) \end{vmatrix}$$

$$= \hat{x}(-12y - 9y) - \hat{y}(-4x - 6x) + \hat{z}(0 - 0)$$

$$= -21y\,\hat{x} + 10x\,\hat{y}$$

$$(\bar{B}\cdot\bar{\nabla})\bar{A} = 3y\,\hat{x} - 4x\,\hat{y} \quad ; \quad (\bar{A}\circ\bar{\nabla})\bar{B} = 6y\,\hat{x} - 2x\,\hat{y}$$

$$\bar{A}(\bar{\nabla}\circ\bar{B}) = (x\hat{x} + 2y\hat{y} + 3z\hat{z})\left(\frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(-2x)\right) = 0$$

$$\bar{B}(\bar{\nabla}\cdot\bar{A}) = (3y\,\hat{x} - 2x\,\hat{y})(1 + 2 + 3) = 18y\,\hat{x} - 12x\,\hat{y}$$

$$\therefore \text{RHS} = (\bar{B}.\bar{\nabla})\bar{A} - (\bar{A}.\bar{\nabla})\bar{B} + \bar{A}(\bar{\nabla}\circ\bar{B}) - \bar{B}(\bar{\nabla}\circ\bar{A})$$

$$= 3y\hat{x} - 4x\hat{y} - 6y\hat{x} + 2x\hat{y} + 0 - 18y\hat{x} + 12x\hat{y}$$

$$= (3y - 6y - 18y)\hat{x} + (-4x + 2x + 12x)\hat{y}$$

$$= -21y\hat{x} + 10x\hat{y}$$

Thus. LHS = RHS

Problem 1.26

Problem 1.26 Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$.

(b) $T_b = \sin x \sin y \sin z$.

(c) $T_c = e^{-5x} \sin 4y \cos 3z$.

(d) $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

Solution :-

a) $$\nabla^2 T_a = \frac{\partial^2}{\partial x^2}\left(x^2+2xy+3z+4\right) + \frac{\partial^2}{\partial y^2}\left(x^2+2xy+3z+4\right) + \frac{\partial^2}{\partial z^2}\left(x^2+2xy+3z+4\right)$$

$$= 2 + 0 + 0$$
$$= 2$$

b) $$\nabla^2 T_b = \frac{\partial^2}{\partial x^2}\left(\sin x \sin y \sin z\right) + \frac{\partial^2}{\partial y^2}\left(\sin x \sin y \sin z\right) + \frac{\partial^2}{\partial z^2}\left(\sin x \sin y \sin z\right)$$

$$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$
$$= -3 \sin x \sin y \sin z = -3T_b$$

c) $$\nabla^2 T_c = \frac{\partial^2}{\partial x^2}\left(e^{-5x} \sin 4y \cos 4z\right) + \frac{\partial^2}{\partial y^2}\left(e^{-5x} \sin 4y \cos 4z\right) + \frac{\partial^2}{\partial z^2}\left(e^{-5x} \sin 4y \cos 4z\right)$$

$$= (-5)(-5)\, e^{-5x} \sin 4y \cos 3z \, +$$
$$+ \, 4(-4)\, e^{-5x} \sin 4y \cos 3z \, +$$
$$(-4)4\, e^{-5x} \sin 4y \cos 3z$$

$$= (25 - 16 - 9)\, T_e$$

$$= 0$$

d) $\bar{v} = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z}$

$$\nabla^2 v = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$$

$$= (2)\hat{x} + (6x)\hat{y} + 0$$

$$= 2\hat{x} + 6x\hat{y}$$

Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v} \cdot d\mathbf{l}$ for the loop that goes from **a** to **b** along (1) and returns to **a** along (2)?

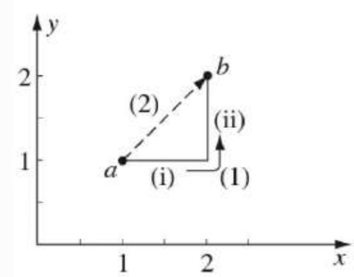


FIGURE 1.21

Solution :-

Along path (1).

$$\int_1 \bar{v}\cdot d\bar{l} = \int_1^2 \left[y^2\hat{x} + 2x(y+1)\,\hat{y}\right]\cdot(dx\,\hat{x})$$

$$+ \int_1^2 \left[y^2\hat{x} + 2x(y+1)\,\hat{y}\right]\cdot\left[dy\,\hat{y}\right]$$

$$= \int_1^2 y^2\,dx + \int_1^2 2x(y+1)\,dy$$

$$= y^2 x\Big|_1^2 + 2x\left[\frac{y^2}{2} + y\right]_1^2$$

$$= 1^2.(2-1) + 2.2\left[\frac{2^2}{2} + 2 - \frac{1^2}{2} - 1\right]$$

$$= 1 + 4\left[\frac{4}{2} + \frac{4}{2} - \frac{1}{2} - \frac{2}{2}\right]$$

$$= 1 + 2[5]$$

$$= 11$$

Along path 2.

$$y = \frac{2-1}{2-1}x \Rightarrow y = x \Rightarrow dy = dx$$

$$\int_2 \bar{v}\cdot d\bar{l} = \int\left[y^2\hat{x} + 2x(y+1)\hat{y}\right]\left[dx\,\hat{x} + dy\,\hat{y}\right]$$

$$= \int\left(y^2dx + 2x(y+1)\,dy\right)$$

$$= \int_1^2 x^2dx + 2x(x+1)\,dx$$

$$= \int_1^2 (3x^2 + 2x)\,dx$$

$$= \left[3\cdot\frac{x^3}{3} + 2\cdot\frac{x^2}{2}\right]_1^2$$

$$= \left[8-1+4-1\right]$$

$$= 10.$$

$$\oint_{1\to 2} \bar{v}\cdot d\bar{l} = 11 - 10 = 1$$

//

Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.

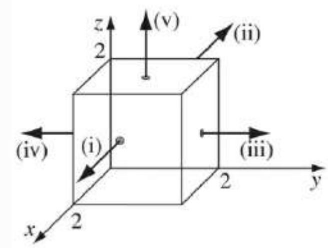


FIGURE 1.23

Solution :-

for surface (i)

$$d\bar{A} = dy\,dz\,\hat{x}$$

$$\int_S \bar{v}\cdot d\bar{A} = \int_S 2xz\,dy\,dz$$

$$= 2x\int_0^2 dy\int_0^2 z\,dz$$

$$= 2.2\,\left[y\right]_0^2\left[\frac{z^2}{2}\right]_0^2$$

$$= 4\cdot 2\cdot \frac{2^2}{2}$$

$$= 16$$

for surface (ii) $\qquad d\bar{A} = dy\,dz\,(-\hat{x})$

$x=0$.

$$\bar{v}\cdot d\bar{A} = -2xz\,dy\,dz\Big|_{x=0} = 0.$$

$$\therefore \int \bar{v}\cdot d\bar{A} = 0$$

for surface (iii) $\qquad d\bar{A} = dx\,dz\,\hat{y}$

$$\therefore \int_S \bar{v}\cdot d\bar{A} = \int_S (x+2)\,dx\,dz = \int_0^2 dz\left(\int_0^2 (x+2)\,dx\right)$$

$$= \int_0^2 dz\left[\frac{x^2}{2} + 2x\right]_0^2 = 6\,[2-0] = 12$$

for surface (R) $d\bar{A} = dx\,dz\,-\hat{y}$

$$\int_S \bar{v}\cdot d\bar{A} = -\int_S (x+2)\,dx\,dz = -12$$

for surface (V) $dA = dx\,dy\,\hat{z}$

$$\int \bar{v}\cdot d\bar{A} = \int_S y\,(z^2-3)\,dx\,dy$$

$$= (2^2-3)\int_0^2 y\left\{\int_0^2 dx\right\}dy$$

$$= 1\cdot 2\cdot \frac{y^2}{2}\Big|_0^2 = 2\times 2 = 4$$

$\therefore$ total surface integral

$$\int_S \bar{v}\cdot d\bar{A} = 16+0+12-12+4 = 20$$

Example 1.8. Calculate the volume integral of $T = xyz^2$ over the prism in Fig. 1.24.

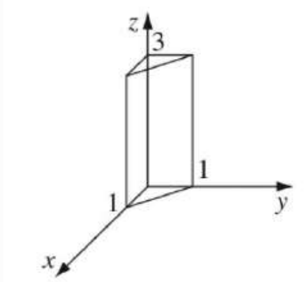
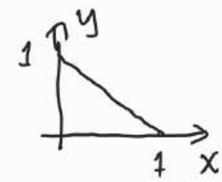


FIGURE 1.24

Solution :-

$$\int_V T \cdot d\tau = \int dz \int dy \int dx \left(xyz^2\right)$$

$$= \int dz \int dy \int_0^{1-y} yz^2 x \, dx$$



$$= \int dz \int dy \left(yz^2 \frac{x^2}{2} \Big|_0^{1-y} \right)$$

$y = -1x + 1$

$\Rightarrow x = 1 - y$

$$= \int dz \int_0^1 dy \;\; z^2 \left(y \frac{(1-y)^2}{2} \right)$$

$$= \int dz \frac{z^2}{2} \int_0^1 dy \left(y - 2y^2 + y^3 \right)$$

$$= \int dz \cdot \frac{z^2}{2} \left(\frac{y^2}{2} - 2 \cdot \frac{y^3}{3} + \frac{y^4}{4} \right)_0^1$$

$$= \int dz \frac{z^2}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right)$$

$$= \frac{1}{2} \left(\frac{6 - 8 + 3}{12} \right) \frac{z^3}{3} \Big|_0^3$$

$$= \frac{1}{2} \cdot \frac{1}{12} \cdot \frac{\cancel{27}}{\cancel{3}} \, 9$$

$$= \frac{3}{2 \cdot 4} = \frac{3}{8} \, //$$

Problem 1.29 Calculate the line integral of the function $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + y^2\,\hat{\mathbf{z}}$ from the origin to the point (1,1,1) by three different routes:

(a) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$.

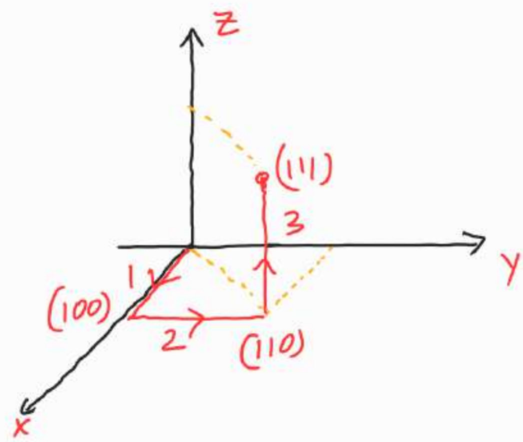
(b) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$.

(c) The direct straight line.

(d) What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b)?

Solution :-

a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$



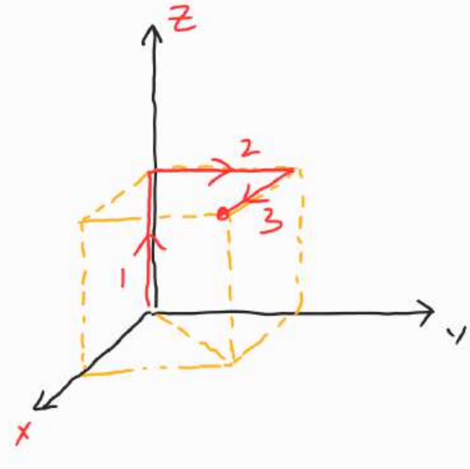
$$\int_c \bar{v}\cdot d\bar{l} = \int_1 \bar{v}\cdot dx\hat{x} + \int_2 \bar{v}\cdot dy\hat{y} + \int_3 \bar{v}\cdot dz\hat{z}$$

$$= \int_0^1 x^2 dx + \int_0^1 2yz\, dy + \int_0^1 y^2 dz$$

$$= \frac{x^3}{3}\Big|_0^1 + 2.0\int y\,dy + 1^2\int_0^1 dz$$

$$= \frac{1}{3} + 0 + 1 = \frac{4}{3}$$

b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$



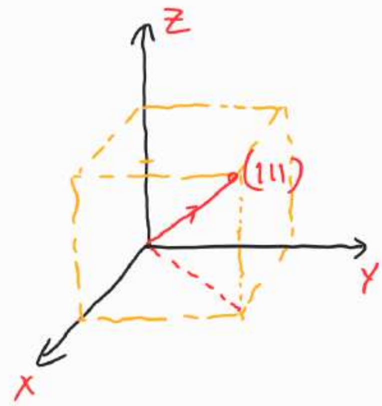
$$\int_c \bar{v}\cdot d\bar{l} = \int_1 \bar{v}\cdot dz\hat{z} + \int_2 \bar{v}\cdot dy\hat{y} + \int_3 \bar{v}\cdot dx\hat{x}$$

$$= \int_0^1 y^2\, dz + \int_0^1 2yz\, dy + \int_0^1 x^2 dx$$

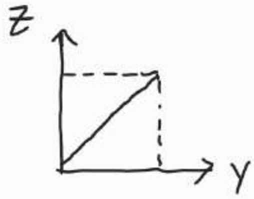
$$= 0^2\int_0^1 dz + 2.1\ \frac{y^2}{2}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1$$

$$= 0 + 2 + \frac{1}{3} = \frac{7}{3}.$$

c) the straight line



$$\int \bar{v}\cdot d\bar{l} = \int_0^1 x^2 dx + \int_0^1 2yz\,dy + \int_0^1 y^2 dz$$



$$z = y + 0 \Rightarrow z = y \Rightarrow dz = dy$$

$$\int \bar{v}\cdot d\bar{l} = \frac{x^3}{3}\Big|_0^1 + \int_0^1 2y^2 dy + \int_0^1 z^2 dz$$

$$= \frac{1}{3} + 2\,\frac{y^3}{3}\Big|_0^1 + \frac{z^3}{3}\Big|_0^1$$

$$= \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{4}{3}$$

//

d) along path (a) and back on path (b)

$$\int \bar{v}\cdot d\bar{l} = \frac{4}{3} - \frac{7}{3} = -\frac{3}{3} = -1$$

//

Example 1.9. Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

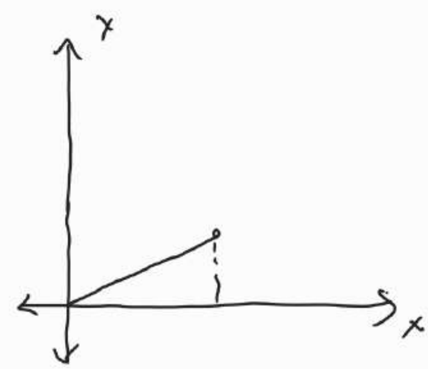
Solution :

To check $\int \nabla T \cdot dl = T(b) - T(a)$.

$$\text{LHS} = \int \nabla T \cdot dl$$

$$= \int \left\{ \frac{\partial}{\partial x}(xy^2)\hat{x} + \frac{\partial}{\partial y}(xy^2)\hat{y} + \frac{\partial}{\partial z}(xy^2)\hat{z} \right\} \cdot \left\{ dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \right\}$$

$$= \int \left(y^2 dx + 2xy\,dy \right)$$



$$y = \frac{1}{2}x$$

$$\Rightarrow 2dy = dx.$$

$$= \int \left(\frac{1}{4}x^2 dx + x \cdot \frac{1}{2}x \cdot dx \right)$$

$$= \int_0^2 \left(\frac{x^2}{4} + \frac{x^2}{2} \right) dx$$

$$= \frac{3}{4} \frac{x^3}{3} \Big|_0^2$$

$$= \frac{8}{4} = 2 \ //.$$

$$\text{RHS} = xy^2 \Big|_{x=2, y=1} - xy^2 \Big|_{x=0, y=0}$$

$$= 2$$

LHS = RHS //

Problem 1.30 Calculate the surface integral of the function in Ex. 1.7, over the *bottom* of the box. For consistency, let "upward" be the positive direction. Does the surface integral depend only on the boundary line for this function? What is the total flux over the *closed* surface of the box (*including* the bottom)? [*Note:* For the *closed* surface, the positive direction is "outward," and hence "down," for the bottom face.]

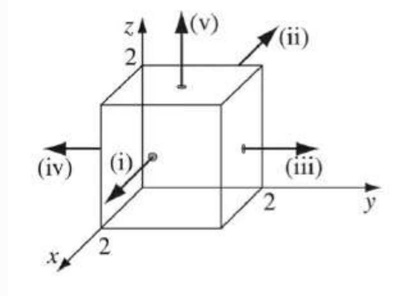


FIGURE 1.23

$$d\bar{A} = -dx\,dy\,\hat{z}$$

$$\int \bar{v} \cdot d\bar{A} = -\int y(z^2-3)\,dx\,dy$$

$$= -(0^2-3)\int_0^2 dx \int_0^2 y\,dy$$

$$= 3 \cdot 2 \cdot \frac{4}{2} = 12$$

does not depend on boundary line.

previously we got flux = 20.

total flux = 20 + 12 = 32

Problem 1.32 Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = (1, 1, 1)$, and the three paths in Fig. 1.28:

(a) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$;

(b) $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$;

(c) the parabolic path $z = x^2$; $y = x$.

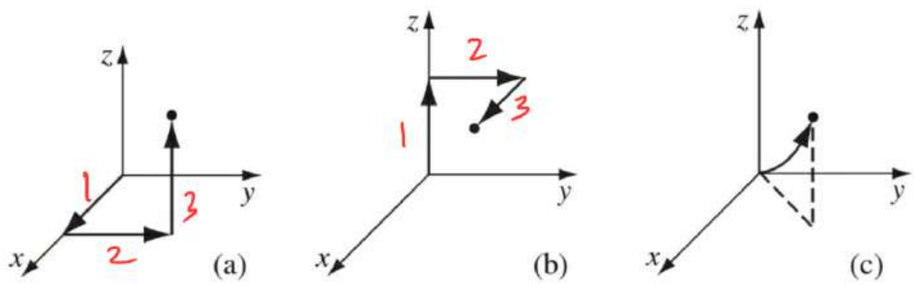


FIGURE 1.28

Solution :- a) $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$

$$\bar{\nabla} T = (2x+4y)\hat{x} + (4x+2z^3)\hat{y} + (6yz^2)\hat{z}$$

$$\int_l \bar{\nabla}T \cdot d\bar{l} = \int_1 \bar{\nabla}T \cdot dx\,\hat{x} + \int_2 \bar{\nabla}T \cdot dy\,\hat{y} + \int_3 \bar{\nabla}T \cdot dz\,\hat{z}$$

$$= \int_0^1 2x\,dx + \int_0^1 4x\,dy + \int_0^1 6y\,z^2\,dz$$

$$= x^2\Big|_0^1 + 4.1(1-0) + 6.1.\frac{z^3}{3}\Big|_0^1$$

$$= 1+4+2 = 7$$

$$T(1,1,1) - T(0,0,0) = 1+4+2-(0+0+0)$$
$$= 7$$

Thus, fundamental theorem on gradients is satisfied.

b). $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$.

$$\bar{\nabla} T = (2x+4y)\hat{x} + (4x+2z^3)\hat{y} + (6yz^2)\hat{z}$$

$$\int_l \bar{\nabla}T \cdot d\bar{l} = \int_1 \bar{\nabla}T \cdot dz\,\hat{z} + \int_2 \bar{\nabla}T \cdot dy\,\hat{y} + \int_3 \bar{\nabla}T \cdot dx\,\hat{x}$$

$$= 6 \cdot 0 \int z^2 dz + 2 \cdot 1^3 \int_0^1 dy + \int_0^1 (2x+4 \cdot 1)\,dx$$

$$= 0 + 2(1-0) + (x^2+4x)\Big|_0^1$$

$$= 2 + 1 + 4 = 7$$

Thus it satisfies the fundamental theorem for gradients.

c) the parabolic path $z = x^2$; $y = x$

$$\nabla \bar{T} \cdot d\bar{l} = (2x+4y)\,dx + (4x+2z^3)\,dy + (6yz^2)\,dz.$$

$$y = x \Rightarrow dx = dy.$$

$$\Rightarrow \bar{\nabla}T \cdot d\bar{l} = 6x\,dx + (4x+2z^3)\,dx + 6xz^2\,dz$$

$$= (10x + 2z^3)\,dx + 6xz^2\,dz$$

$z = x^2 \Rightarrow dz = 2x\,dx.$

$$\therefore \bar{\nabla}T \cdot d\bar{l} = (10x + 2 \cdot x^6)\,dx + 12x^2 \cdot x^4$$

$$= (10x + 14x^6)\,dx.$$

$$\int \bar{\nabla}T \cdot d\bar{l} = \int_0^1 (10x + 14x^6)\,dx$$

$$= 10\frac{x^2}{2}\Big|_0^1 + 14 \cdot \frac{x^7}{7}\Big|_0^1$$

$$= 5 + 2$$

$$= 7$$

Thus, it satisfies the fundamental theorem for gradients.

//

Example 1.10. Check the divergence theorem using the function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

and a unit cube at the origin (Fig. 1.29).

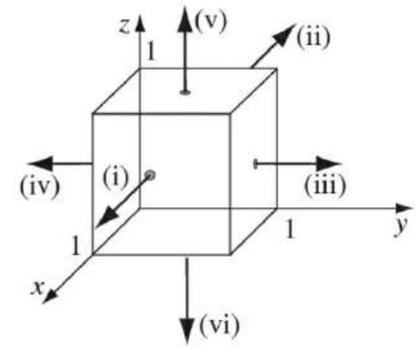


FIGURE 1.29

Solution :-

$$\int (\nabla \cdot v)\, d\tau = \oint \bar{v} \cdot d\bar{A}$$

$$\nabla \cdot v = 2x + 2y$$

$$\int_V (\nabla \cdot v)\, d\tau = \int_0^1\int_0^1\int_0^1 (2x + 2y)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 \left\{ (2yx + x^2)\Big|_0^1 \right\} dy\, dz$$

$$= \int_0^1\int_0^1 (2y + 1)\, dy\, dz$$

$$= \int_0^1 (y^2 + y)\Big|_0^1 dz$$

$$= 2\int_0^1 dz = 2$$

$$\int \bar{v} \cdot d\bar{a}_i - \int_0^1\int_0^1 y^2\, dy\, dz = \int_0^1 \frac{y^3}{3}\Big|_0^1 dz = \frac{1}{3}\, z\Big|_0^1 = \frac{1}{3}$$

$$\int \bar{v} \cdot d\bar{a}_{ii} = -\int_0^1\int_0^1 y^2\, dy\, dz = -\frac{1}{3}$$

$$\int \bar{v} \cdot d\bar{a}_{iii} = \int_0^1 \int_0^1 (2xy + z^2)\, dx\, dz = \int_0^1 \int_0^1 (2x + z^2)\, dx\, dz$$

$$= \int_0^1 \left(2 \cdot \frac{x^2}{2} + z^2 (x) \Big|_0^1 \right) dz$$

$$= \int_0^1 (1 + z^2)\, dz$$

$$= z + \frac{z^3}{3} \Big|_0^1 = \frac{4}{3}.$$

$$\int \bar{v} \cdot d\bar{a}_{iv} = -\int_0^1 \int_0^1 (2xy + z^2)\, dx\, dz = -\int_0^1 \int_0^1 z^2\, dx\, dz = -\int_0^1 \frac{z^3}{3} \Big|_0^1 dx$$

$$= -\frac{1}{3} x \Big|_0^1$$

$$= -\frac{1}{3}$$

$$\int \bar{v} \cdot d\bar{a}_{v} = \int_0^1 \int_0^1 2yz\, dx\, dy = 2 \int_0^1 \int_0^1 y\, dy\, dx = 2 \int \frac{y^2}{2} \Big|_0^1 dx$$

$$= 2 \cdot \frac{1}{2}\, x \Big|_0^1$$

$$= 1$$

$$\int \bar{v} \cdot d\bar{a}_{vi} = -\int_0^1 \int_0^1 (2yz)\, dx\, dy = 0$$

$$\therefore \oint \bar{v} \cdot d\bar{a} = 0 + 1 - \frac{1}{3} + \frac{4}{3} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} = 1 + \frac{3}{3} = 2$$

Checked. //

Problem 1.33 Test the divergence theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$. Take as your volume the cube shown in Fig. 1.30, with sides of length 2.

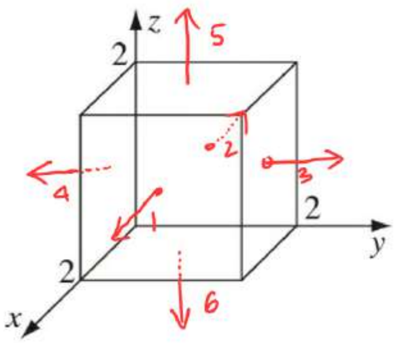


FIGURE 1.30

Solution :- $\int (\bar{\nabla}\cdot\bar{v})\,d\tau = \oint \bar{v}\cdot d\bar{A}$

$$\bar{\nabla}\cdot\bar{v} = y + 2z + 3x$$

$$\int_V \bar{\nabla}\cdot\bar{v}\,d\tau = \int_0^2\int_0^2\int_0^2 (y + 2z + 3x)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 \left\{ yx + 2zx + 3\frac{x^2}{2}\Big|_0^2 \right\} dy\,dz$$

$$= \int_0^2\int_0^2 (2y + 4z + 6)\,dy\,dz$$

$$= \int_0^2 \left\{ 2\frac{y^2}{2} + 4zy + 6y\Big|_0^2 \right\} dz$$

$$= \int_0^2 (4 + 8z + 12)\,dz$$

$$= 16z + 8\frac{z^2}{2}\Big|_0^2$$

$$= 32 + 16 = 48$$

$$\int \bar{v}\cdot d\bar{A}_1 = \int_0^2\int_0^2 xy \, dy\,dz = \int_0^2 \left\{ 2\,\frac{y^2}{2}\Big|_0^2 \right\} dz = 4z\Big|_0^2 = 8$$

$$\int \bar{v}\cdot d\bar{A}_2 = -\int_0^2\int_0^2 xy \, dy\,dz = 0$$

$$\int \bar{v}\cdot d\bar{A}_3 = \int_0^2\int_0^2 2yz \, dx\,dz = \int_0^2 4\,\frac{z^2}{2}\Big|_0^2 dx = 8x\Big|_0^2 = 16$$

$$\int \bar{v}\cdot d\bar{A}_4 = -\int_0^2\int_0^2 2yz \, dx\,dz = 0$$

$$\int \bar{v}\cdot d\bar{A}_5 = \int_0^2\int_0^2 3zx \, dy\,dx = \int \left\{ 6\,\frac{x^2}{2}\Big|_0^2 \right\} dy = 12y\Big|_0^2 = 24$$

$$\int \bar{v}\cdot d\bar{A}_6 = -\int_0^2\int_0^2 3zx \, dy\,dx = 0$$

$$\therefore \oint \bar{v}\cdot d\bar{A} = 0 + 24 + 0 + 16 + 8 + 0 = 48$$

//

Example 1.11. Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$. Check Stokes' theorem for the square surface shown in Fig. 1.33.

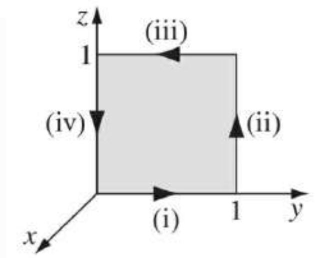


FIGURE 1.33

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{A} = \oint \bar{v} \cdot d\bar{l}$$

$$\bar{\nabla} \times \bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & (2xz+3y^2) & (4yz^2) \end{vmatrix}$$

$$= \hat{x}\left(4z^2 - 2x\right) - \hat{y}\left(0-0\right) + \hat{z}\left(2z - 0\right)$$

$$= (4z^2 - 2x)\hat{x} + 2z\hat{z}$$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{A} = \int_0^1\int_0^1 (4z^2 - 2x)\, dy\, dz = \int_0^1\int_0^1 4z^2\, dy\, dz$$

$$= \int 4\frac{z^3}{3}\Big|_0^1 dy$$

$$= \frac{4}{3} y\Big|_0^1$$

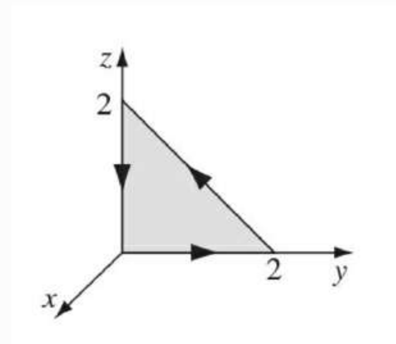
$$= \frac{4}{3}$$

$$\oint \bar{v} \cdot d\bar{l} = \int_0^1 3y^2 dy + \int_0^1 4.1.z^2 dz + \int_1^0 3y^2 (dy) + \int_1^0 (-0.z^2) dz$$

$$= y^3\Big|_0^1 + \frac{4}{3}z^3\Big|_0^1 - y^3\Big|_1^0 + 0$$

$$= 1 + \frac{4}{3} - 1 = 4/3$$

Problem 1.34 Test Stokes' theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$, using the triangular shaded area of Fig. 1.34.



Solution:-

$$(\bar{\nabla}\times\bar{v}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= \hat{x}\,(0 - 2y) - \hat{y}\,(3z - 0) + \hat{z}\,(0 - x)$$

$$= -2y\hat{x} - 3z\hat{y} - x\hat{z}$$

$$\int(\bar{\nabla}\times\bar{v})\cdot d\bar{A} = \int_0^2\int_0^{z-2} -2y\;dy\,dz$$

$z = -1y + 2$

$\Rightarrow y = z - 2$

$\Rightarrow dy = dz.$

$$= -\int_0^2 y^2\Big|_0^{z-2} dz$$

$$= -\int_0^2 (z-2)^2\,dz$$

$$= -\left\{\frac{z^3}{3}\Big|_0^2 - 4\frac{z^2}{2}\Big|_0^2 + 4z\Big|_0^2\right\}$$

$$= -\left\{\frac{8}{3} - \frac{16}{2} + 8\right\}$$

$$= -\frac{8}{3}$$

$$\oint \bar{v} \cdot d\bar{l} = \int_0^2 (2yz)\,dy + \int_0^2 \left(2yz\,dy + 3zx\,dz\right) + \int_2^0 3zx\,(-dz)$$

$$= 0 + \int_0^2 \left(2(z-2)z + 3zx\right) dz - 0$$

$$= \int_0^2 2(z^2 - 2z)\,dz$$

$$= 2\left[\frac{z^3}{3} - 2\cdot\frac{z^2}{2}\right]_0^2$$

$$= 2\left[\frac{8}{3} - 4\right]$$

$$= 2\left[\frac{8-12}{3}\right]$$

$$= -\frac{8}{3}$$

//

Problem 1.35 Check Corollary 1 by using the same function and boundary line as in Ex. 1.11, but integrating over the five faces of the cube in Fig. 1.35. The back of the cube is open.

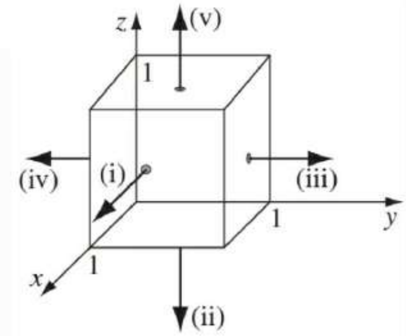


FIGURE 1.35

Solution: $\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}$ depends only on the boundary line, not on the particular surface used

$$\bar{\nabla} \times \bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & (2xz+3y^2) & (4yz^2) \end{vmatrix}$$

$$= \hat{x}(4z^2 - 2x) - \hat{y}(0-0) + \hat{z}(2z - 0)$$

$$= (4z^2 - 2x)\hat{x} + 2z\hat{z}$$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}_i = \iint (4z^2 - 2x)\, dy\, dz$$

$$= \iint (4z^2 - 1)\, dy\, dz$$

$$= \int_0^1 \int_0^1 \left(4 \cdot \frac{z^3}{3} - 2z\right) dy$$

$$= \int_0^1 \left(\frac{4}{3} - 2\right) dy = -\frac{2}{3} \cdot y \Big|_0^1 = -\frac{2}{3}$$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}_{ii} = -\iint (2z)\, dx\, dy = 0$$

$$\int (\bar{\nabla} \times \bar{v})\, d\bar{a}_{iii} = \iint 0 \cdot dx\, dz = 0$$

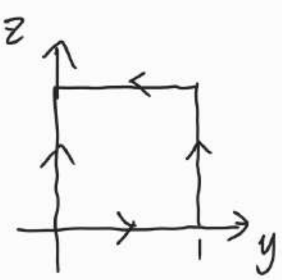
$$\int (\bar{\nabla} \times \bar{v})\, d\bar{a}_{iv} = 0$$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}_{v} = \iint 2z \cdot dx\, dy$$

$$= 2 \int_0^1 \int_0^1 dx\, dy$$

$$= 2$$

$$\int (\bar{\nabla} \times \bar{v})\, d\bar{A} = 2 - \frac{2}{3} = \frac{6-2}{3} = \frac{4}{3}$$



$$\int \bar{v} \cdot d\bar{l} = \int_0^1 3y^2\, dy + \int_0^1 (4z^2)\, dz$$

$$+ \int_1^0 3y^2\, dy - \int_1^0 (0)\, dz$$

$$= 3\frac{y^3}{3}\Big|_0^1 + 4\frac{z^3}{3}\Big|_0^1 -$$

$$+ 3\frac{y^3}{3}\Big|_1^0 - 0$$

$$= y^3\Big|_0^1 + \frac{4}{3} - y^3\Big|_1^0$$

$$= 1 - 0 + \frac{4}{3} + (0-1)$$

$$= \frac{4}{3}$$

Example 1.13. Find the volume of a sphere of radius R.

Solution

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\therefore \quad V = \int_0^r \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^r r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^\pi d\phi$$

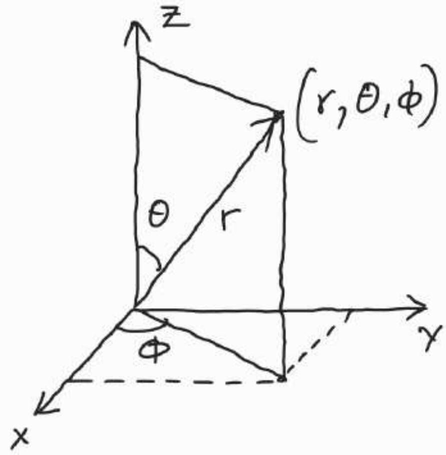
$$= \frac{r^3}{3}\Big|_0^r \left(-\cos\theta\Big|_0^\pi\right) \phi\Big|_0^{2\pi}$$

$$= \frac{r^3}{3} \left\{-(-1-1)\right\} \cdot 2\pi$$

$$= \frac{4}{3}\pi r^3$$

//

Problem 1.37 Find formulas for r, θ, ϕ in terms of x, y, z (the inverse, in other words, of Eq. 1.62).



Solution :-

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$x^2+y^2+z^2 = r^2\left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right)$$

$$= r^2\left(\sin^2\theta\left(\cos^2\phi+\sin^2\phi\right)+\cos^2\theta\right)$$
$$= r^2\left(\sin^2\theta+\cos^2\theta\right)$$
$$= r^2$$

$$\therefore\ r = \sqrt{x^2+y^2+z^2}$$

Again, $x^2+y^2 = r^2\left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right)$

$$\Rightarrow x^2+y^2 = r^2\sin^2\theta$$
$$\Rightarrow \sin\theta = \sqrt{\frac{x^2+y^2}{r^2}}$$
$$\Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{r}\right)$$

Again, $y/x = \tan\phi$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Problem 1.39

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

(b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

$$\bar{v}_1 = r^2\hat{r}$$

$$\bar{\nabla}\cdot\bar{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4\right)$$

$$= \frac{4r^3}{r^2} = 4r$$

$$\int(\bar{\nabla}\cdot\bar{v}_1)\,d\tau = \iiint 4r\;r^2 dr\sin\theta\, d\theta\, d\phi$$

$$= 4\pi\int_0^R 4r^3\,dr$$

$$= 4\pi\left(\cancel{4}\,\frac{r^4}{\cancel{4}}\right)_0^R = 4\pi R^4$$

$$\oint_s \bar{v}_1\cdot d\bar{A} = \int_0^{\pi}\int_0^{2\pi} r^2\hat{r}\;\;\hat{r}\,r^2\sin\theta\, d\theta\, d\phi$$

$$= 4\pi R^4$$

b) $\bar{v}_2 = \left(\frac{1}{r^2}\right)\hat{r}$

$$\bar{\nabla} \cdot \bar{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right)$$

$$= 0$$

$$\int (\bar{\nabla} \cdot \bar{v}_2)\, d\tau = 0$$

$$\oint \bar{v}_2 \cdot d\bar{A} = \int_0^{\pi}\int_0^{2\pi} \frac{1}{r^2} \cdot r^2 \sin\theta \, d\theta \, d\phi$$

$$= 4\pi \quad !!!$$

Problem 1.40 Compute the divergence of the function

$$\mathbf{v} = (r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}.$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin (Fig. 1.40).

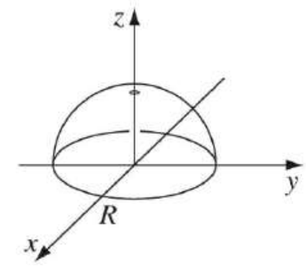


FIGURE 1.40

Solution :-

$$\bar{\nabla}\cdot\bar{v} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\, r\cos\theta\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\, r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(r\, r\sin\theta\cos\phi\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\cos\theta\, 3r^2 + r^2\, 2\sin\theta\cos\theta - r^2\sin\theta\sin\phi\right]$$

$$= 5\cos\theta - \sin\phi$$

DIVERGENCE THEOREM : $\int(\bar{\nabla}\cdot\bar{v})\,d\tau = \oint \bar{v}\cdot d\bar{A}$

$$\int(\bar{\nabla}\cdot\bar{v})\,d\tau = \int_0^R\int_0^{\pi/2}\int_0^{2\pi}\left(5\cos\theta - \sin\phi\right) r^2\,dr\,\sin\theta\,d\theta\,d\phi$$

$$= \frac{R^3}{3}\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi}\left(5\cos\theta\sin\theta - \sin\phi\sin\theta\right)d\phi$$

$$= \frac{R^3}{3}\int_0^{\pi/2} d\theta \left(5\cos\theta\sin\theta\,.\,\phi + \cos\phi\sin\theta\right)\Big|_{\phi=0}^{2\pi}$$

$$= \frac{R^3}{3}\int_0^{\pi/2} d\theta \left(10\pi \cos\theta \sin\theta \right)$$

$$= 10\pi \frac{R^3}{3} \cdot \frac{1}{2} \cdot \int_0^{\pi/2} 2\cos\theta \sin\theta$$

$$= 5\pi \frac{R^3}{3} \int_0^{\pi/2} \sin(2\theta)\, d\theta$$

$$= 5\pi \frac{R^3}{3} \left\{ -\frac{\cos 2\theta}{2} \Big|_0^{\pi/2} \right\}$$

$$= \frac{5\pi R^3}{6} \left(-\cos\pi + \cos 0 \right)$$

$$= \frac{5\pi R^3}{6} \cdot 2$$

$$= \frac{5}{3}\pi R^3$$

Again. $\oint \bar{v} \cdot d\bar{A} = \int_{\text{curved}} \bar{v} \cdot d\bar{A}_1 + \int_{\text{plane}} \bar{v} \cdot d\bar{A}_2$

$$= \iint r\cos\theta \, r^2 \sin\theta \, d\theta \, d\phi + \iint r\sin\theta \, r \, dr \sin\theta \, d\phi$$

$$= R^3 \iint \cos\theta \sin\theta \, d\theta \, d\phi + \frac{r^3}{3}\Big|_0^R \sin^2\frac{\pi}{2} \cdot 2\pi$$

$$= 2\pi R^3 \cdot \frac{1}{2} \int_0^{\pi/2} 2\cos\theta \sin\theta \, d\theta + 2\pi \frac{R^3}{3}$$

$$= \pi R^3 \cdot \int_0^{\pi/2} \sin 2\theta + 2\pi \frac{R^3}{3}$$

$$= \pi R^3 \cdot 1 + 2\pi \frac{R^3}{3} = \frac{5}{3}\pi R^3 \quad //$$

Problem 1.43

(a) Find the divergence of the function

$$\mathbf{v} = s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\;\hat{\boldsymbol{\phi}} + 3z\;\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

Solution : $\frac{1}{s}\left[\frac{\partial}{\partial s} s^2\left(2+\sin^2\phi\right) + \frac{\partial}{\partial \phi} s\sin\phi\cos\phi + \frac{\partial}{\partial z} 3sz\right]$

$$= \frac{1}{s}\left[2s\left(2+\sin^2\phi\right) + s\cos^2\phi - s\sin^2\phi + 3s\right]$$

$$= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3$$

$$= 7 + \sin^2\phi + \cos^2\phi$$

$$= 7+1 = 8 \quad //$$

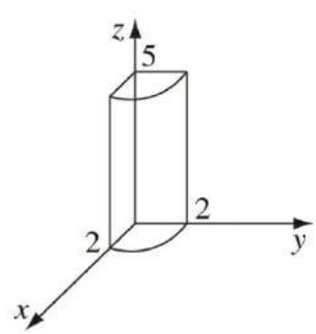


FIGURE 1.43

Divergence Theorem : $\int(\bar{\nabla}\cdot\bar{v})\,d\tau = \oint \bar{v}\cdot d\bar{a}$

$$\int(\bar{\nabla}\cdot\bar{v})\,d\tau = \iiint 8.\ s\,ds\,d\phi\,dz$$

$$= 8\ \frac{s^2}{2}\Big|_0^2\ \phi\Big|_0^{\pi/2}\cdot z\Big|_0^5$$

$$= 8\cdot\frac{4}{2}\cdot\frac{\pi}{2}\cdot 5 = 40\pi$$

$$\int \bar{v}\cdot d\bar{a} = \int_1 \bar{v}\cdot\hat{s}\,s\,d\phi\,dz + \int_2 \bar{v}\cdot\hat{z}\,s\,d\phi\,ds + \int_3 \bar{v}\cdot\left(-\hat{z}\,s\,d\phi\,ds\right)$$

$$+ \int_4 \bar{v}\cdot\hat{\phi}\,ds\,dz + \int_5 v\cdot\left(-\hat{\phi}\,ds\,dz\right)$$

$$= \iint s^2(2+\sin^2\phi)\,d\phi\,dz + \iint 3zs\,d\phi\,ds - \iint 3zs\,d\phi\,ds$$
$$+ \iint s\sin\phi\cos\phi\,ds\,dz - \iint s\sin\phi\cos\phi\,ds\,dz$$

$$= 2^2 . 5 \int_0^{\pi/2} (2+\sin^2\phi)\,d\phi + 3 . 5 \left.\frac{s^2}{2}\right|_0^2 \phi\Big|_0^{\pi/2}$$
$$- 0 + \sin\frac{\pi}{2}\underbrace{\cancel{\cos}\frac{\pi}{2}}_{0}\iint s\,ds\,dz - \sin\frac{\pi}{2}\underbrace{\cancel{\cos}\frac{\pi}{2}}_{0}\iint s\,ds\,dz$$
$$= 20\int_0^{\pi/2}\left\{2+\frac{1}{2}(1-\cos 2\phi)\right\}d\phi + 15 . 2 . \frac{\pi}{2}$$
$$= 20\int_0^{\pi/2}\left(\frac{5}{2}-\frac{1}{2}\cos 2\phi\right)d\phi + 15\pi$$
$$= 20\left(\frac{5}{2}\phi - \frac{1}{2}.\frac{1}{2}\sin 2\phi\right)_0^{\pi/2} + 15\pi$$
$$= 20 . \frac{5}{2} . \pi/2 + 15\pi$$
$$= 25\pi + 15\pi = 40\pi$$

$$\bar{\nabla}\times\bar{v} = \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ s(2+\sin^2\phi) & s^2\sin\phi\cos\phi & 3z \end{vmatrix}$$

$$= \frac{1}{s}\left[\hat{s}(0) - s\hat{\phi}(0-0) + \hat{z}\left(2s\sin\phi\cos\phi - s2\sin\phi\cos\phi\right)\right]$$
$$= 0$$

Example 1.14. Evaluate the integral

$$\int_0^3 x^3 \delta(x-2)\, dx.$$

Solution:
$$\int_0^3 x^3\, \delta(x-2)\, dx = 2^3 = 8$$

Problem 1.44 Evaluate the following integrals:

(a) $\int_2^6 (3x^2 - 2x - 1)\, \delta(x-3)\, dx$.

(b) $\int_0^5 \cos x\, \delta(x - \pi)\, dx$.

(c) $\int_0^3 x^3 \delta(x+1)\, dx$.

(d) $\int_{-\infty}^{\infty} \ln(x+3)\, \delta(x+2)\, dx$.

a) $$\int_2^6 (3x^2 - 2x - 1)\, \delta(x-3)\, dx.$$

$$= 3 \cdot 3^2 - 2 \cdot 3 - 1 \quad [\text{since } 2 < 3 < 6]$$

$$= 27 - 6 - 1 = 20$$

b) $$\int_0^5 \cos x\, \delta(x-\pi)\, dx.$$

$$= \cos \pi = -1$$

c) $$\int_0^3 x^3 (x+1)\, dx$$

$$= 0 \quad [\text{since } -1 \notin [0,3]]$$

d) $$\int_{-\infty}^{+\infty} \ln(x+3)\, \delta(x+2)\, dx$$

$$= \ln(-2+3) = \ln(1) = 0$$

//

Problem 1.45 Evaluate the following integrals:

(a) $\int_{-2}^{2}(2x+3)\,\delta(3x)\,dx$.

(b) $\int_{0}^{2}(x^3+3x+2)\,\delta(1-x)\,dx$.

(c) $\int_{-1}^{1} 9x^2\delta(3x+1)\,dx$.

(d) $\int_{-\infty}^{a}\delta(x-b)\,dx$.

a) $$\int_{-2}^{2}(2x+3)\,\delta(3x)\,dx$$

$$= \int_{-2}^{2}(2x+3)\,\frac{1}{3}\,\delta(x)\,dx$$

$$= \frac{1}{3}(2\cdot 0+3) = 1$$

b) $$\int_{0}^{2}(x^3+3x+2)\,\delta(1-x)\,dx$$

$$= \int_{0}^{2}(x^3+3x+2)\,\delta(x-1)\,dx$$

$$= 1^3+3\cdot 1+2 = 6$$

c) $$\int_{-1}^{1} 9x^2\,\delta(3x+1)\,dx$$

$$= \int_{-1}^{1} 9x^2\,\delta\left(3\left(x+\frac{1}{3}\right)\right)dx$$

$$= \int_{-1}^{1} 9x^2\cdot\frac{1}{3}\,\delta\left(x+\frac{1}{3}\right)dx$$

$$= 3\cdot x^2\Big|_{-1/3} = 3\cdot\left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

d) $$\int_{-\infty}^{a}\delta(x-b)\,dx = \begin{cases} 1 & \text{if } -\infty < b < a \\ 0 & \text{if } b \notin (-\infty, a] \end{cases}$$

Example 1.16. Evaluate the integral

$$J = \int_{\mathcal{V}} (r^2 + 2)\, \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau,$$

where $\mathcal{V}$ is a sphere[13] of radius R centered at the origin.

Solution :

$$J = \int_V (r^2+2)\, 4\pi\, \delta^{(3)}(\bar{r})\, d\tau$$

$$= 4\pi\,(0+2) = 8\pi$$

Problem 1.47

(a) Write an expression for the volume charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$. Make sure that the volume integral of ρ equals q.

(b) What is the volume charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{a}$?

(c) What is the volume charge density (in spherical coordinates) of a uniform, infinitesimally thin spherical shell of radius R and total charge Q, centered at the origin? [*Beware:* the integral over all space must equal Q.]

a)

$$\rho(\bar{r}') = q\, \delta^{(3)}(\bar{r}-\bar{r}')$$

$$\int \rho(\bar{r}')\, d\tau = \int q\, \delta^3(\bar{r}-\bar{r}')\, d\tau$$

$$= q \int 1 \cdot \delta^3(\bar{r}-\bar{r}')$$

$$= q$$

b)

$$\rho(\bar{r}') = -q\, \delta^{(3)}(\bar{r}) + q\, \delta^{(3)}(\bar{r}-\bar{r}')$$

c)

$$\rho(\bar{r}) = Q\, \delta^{(3)}(\bar{r}-\bar{R}) = kQ\, \delta(r-R)$$

$$\int \rho(r)\, d\tau = kQ \int_0^{\infty} \delta(r-R)\, dr\, r^2\, d\theta\, d\phi$$

$$\Rightarrow \quad Q = kQ\, 4\pi \int_0^{\infty} \delta(r-R)\, r^2\, dr$$

$$\Rightarrow \quad 1 = 4\pi k \cdot R^2$$

$$\Rightarrow \quad k = \frac{1}{4\pi R^2}$$

$$\therefore \quad \rho(\bar{r}) = \frac{Q}{4\pi R^2}\,\delta(r-R)$$

Problem 1.50

(a) Let $\mathbf{F}_1 = x^2\,\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

(b) Show that $\mathbf{F}_3 = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

a) $\quad \bar{\nabla}\cdot\bar{F}_1 = 0 \qquad\qquad \bar{\nabla}\cdot\bar{F}_2 = 1+1+1 = 3$

$$\bar{\nabla}\times\bar{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix}$$

$$= \hat{x}(0-0) + \hat{y}(2x-0) + \hat{z}(0-0)$$

$$= 2x\,\hat{y}$$

$$\bar{\nabla}\times\bar{F}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= 0$$

Curl of a gradient is always 0.

$\therefore$ $\bar{F}_2$ can be written as the gradient of a scalar.

$$\phi = \frac{1}{2}\left(x^2 + y^2 + z^2\right)$$

$$\bar{\nabla}\phi = \frac{1}{2} 2x\,\hat{x} + \frac{1}{2} 2y\,\hat{y} + \frac{1}{2} 2z\,\hat{z}$$

$$= x\hat{x} + y\hat{y} + z\hat{z}$$

$$= \bar{F}_2$$

Divergence of a curl is always 0.

$\therefore$ $\bar{F}_1$ can be written as curl of a vector.

$$\bar{\nabla}\times\bar{A} = \bar{F}_1$$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = x^2\,\hat{z}$$

$$\Rightarrow \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

$$+ \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = x^2\,\hat{z}$$

$$\frac{\partial A_z}{\partial y} = \frac{\partial A_y}{\partial z} \quad ; \quad \frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z}$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$$

Let $A_x = \text{const.}\ ; \ \frac{\partial A_x}{\partial y} = 0 \ ; \ \frac{\partial A_x}{\partial A_z} = 0$

$A_z = \text{const.}\ ; \ \frac{\partial A_z}{\partial x} = 0 \ ; \ \frac{\partial A_z}{\partial y} = 0$

$A_y = \int x^2 dx = \frac{1}{3}x^3 \ ; \ \frac{\partial A_y}{\partial x} = x^2 \ ; \ \frac{\partial A_y}{\partial z} = 0$

$$\therefore \ \bar{A} = \hat{x} + \frac{1}{3}x^3\hat{y} + \hat{z}$$

b) $\bar{F}_3 = yz\hat{x} + zx\hat{y} + xy\hat{z}$

$\bar{\nabla}\cdot\bar{F}_3 = 0 + 0 + 0 = 0$

Hence $\bar{F}_3$ can be written as the curl of a vector

$$\bar{\nabla}\times\bar{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \hat{x}(x-x) - \hat{y}(y-y) + \hat{z}(z-z)$$

$$= 0$$

Thus, $\bar{F}_3$ can be written as the grad. of a scalar.

$$\bar{F}_3 = \nabla\phi \qquad \phi = xyz$$

then $\nabla\phi = yz\hat{x} + zx\hat{y} + xy\hat{z} = \bar{F}_3$

$$\bar{F}_3 = \bar{\nabla}\times\bar{A}$$

$$\Rightarrow yz\hat{x} + zx\hat{y} + xy\hat{z} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz$$

$$\Rightarrow A_z = \frac{1}{2}y^2z + f_1(x,z) \longmapsto (1)$$

$$\frac{\partial A_y}{\partial z} = -yz + \frac{\partial A_z}{\partial y}$$

$$\Rightarrow A_y = -\frac{1}{2}z^2y + f_2(x,y) \longmapsto (2)$$

$$\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = -zx$$

$$\Rightarrow A_z = -\frac{1}{2}zx^2 + f_3(y,z) \longmapsto (3)$$

$$\frac{\partial A_x}{\partial z} = zx + \frac{\partial A_z}{\partial x}$$

$$\Rightarrow A_x = \frac{1}{2}z^2x + f_4(x,y) \longmapsto (4)$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy$$

$$\Rightarrow A_y = \frac{1}{2}x^2y + f_5(y,z) \longmapsto (5)$$

$$\frac{\partial A_x}{\partial y} = -xy + \frac{\partial A_y}{\partial x}$$

$$\Rightarrow A_x = -\frac{1}{2}xy^2 + f_6(x,z) \longmapsto (6)$$

Comparing 1 & 3

$$f_3(y,z) = \frac{1}{2} y^2 z \quad \text{and} \quad f_1(x,z) = -\frac{1}{2} z x^2$$

Comparing 2 and 5

$$f_5(y,z) = -\frac{1}{2} z^2 y \quad \text{and} \quad f_2(x,y) = \frac{1}{2} x^2 y$$

Comparing 4 and 6

$$f_4(x,y) = -\frac{1}{2} x^2 y \quad \text{and} \quad f_6(x,z) = \frac{1}{2} z^2 x$$

$$\therefore \quad A_x = \frac{1}{2} z^2 x - \frac{1}{2} x y^2 \qquad A_y = \frac{1}{2} x^2 y - \frac{1}{2} z^2 y$$

$$A_z = \frac{1}{2} y^2 z - \frac{1}{2} z x^2$$

$$\bar{A} = \frac{1}{2} x (z^2 - y^2) \hat{x} + \frac{1}{2} y (x^2 - z^2) \hat{y} + \frac{1}{2} z (y^2 - x^2)$$

Problem 1.53

(a) Which of the vectors in Problem 1.15 can be expressed as the gradient of a scalar? Find a scalar function that does the job.

(b) Which can be expressed as the curl of a vector? Find such a vector.

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

a) $\bar{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

$\bar{\nabla}.\bar{v}_a = 2x + 0 - 2x = 0$

$$\bar{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\Rightarrow \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) +$$

$$\hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = x^2$$

$$\Rightarrow A_z = x^2y + f_1(z,x) \longrightarrow ①$$

$$\frac{\partial A_y}{\partial z} = -x^2 + \frac{\partial A_z}{\partial y}$$

$$\Rightarrow A_y = -x^2z + f_2(y,x) \longrightarrow ②$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 3xz^2 \qquad \Big| \qquad \frac{\partial A_z}{\partial x} = -3xz^2 + \frac{\partial A_x}{\partial z}$$

$$\Rightarrow A_x = xz^3 + f_3(x,y) \mapsto (3) \qquad \Big| \qquad \Rightarrow A_z = -\frac{3}{2}x^2z^2 + f_4(z,y) \mapsto (4)$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -2xz \qquad \Big| \qquad \frac{\partial A_x}{\partial y} = 2xz + \frac{\partial A_y}{\partial x}$$

$$\Rightarrow A_y = -x^2z + f_5(y,z) \mapsto (5) \qquad \Big| \qquad \Rightarrow A_x = 2xyz + f_6(x,z) \mapsto (6)$$

$A_x = 0$ since $f_3(x,y) \neq 2xyz$

Comparing (5) and (2)

$f_5(y,z) = f_2(y,z) = 0$

$\therefore A_y = -x^2z$

Comparing (1) and (3)

$f_1(z,x) = -\frac{3}{2}x^2z^2$ but $f(z,y) \neq x^2y$

$\therefore A_z = -\frac{3}{2}x^2z^2$

$$\therefore \quad \bar{A} = -x^2 z\,\hat{y} - \frac{3}{2}x^2 z^2\,\hat{z}$$

c) $$\bar{\nabla} \times \bar{v}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix}$$

$$= \hat{x}\left(2z - 2z\right) - \hat{y}\left(0 - 0\right) + \hat{z}\left(2y - 2y\right)$$

$$= 0$$

$$\bar{\nabla}\phi = \bar{v}_c$$

$$\frac{\partial \phi}{\partial x} = y^2 \quad\Big|\quad \frac{\partial \phi}{\partial y} = 2xy + z^2 \quad\Big|\quad \frac{\partial \phi}{\partial z} = 2yz$$

$$\Rightarrow \phi = xy^2 + f_1(y, z) \quad\Big|\quad \phi = xy^2 + yz^2 + f_2(x, z) \quad\Big|\quad \phi = yz^2 + f(x, y)$$

$$f_2(x, z) = 0$$

$$\phi = xy^2 + yz^2$$

Problem 1.54 Check the divergence theorem for the function

$$\mathbf{v} = r^2 \cos\theta\, \hat{\mathbf{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}},$$

using as your volume one octant of the sphere of radius R (Fig. 1.48). Make sure you include the *entire* surface. [*Answer:* $\pi R^4/4$]

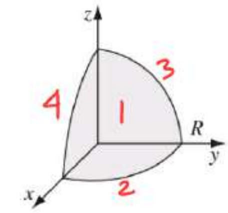


FIGURE 1.48

Solution:

$$\int_V (\bar{\nabla}.\bar{v})\, d\tau = \oint \bar{v}.d\bar{A}$$

$$\bar{\nabla}.\bar{v} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\, r^2\cos\theta\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\, r^2\cos\phi\right) - \frac{\partial}{\partial\phi}\left(r r^2\cos\theta\sin\phi\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\cos\theta\, 4r^3 + r^3\cos\phi\cos\theta - r^3\cos\theta\cos\phi\right]$$

$$= \frac{4r^3\sin\theta\cos\theta}{r^2\sin\theta} = 4r\cos\theta$$

$$\int_V (\bar{\nabla}.\bar{v})\, d\tau = \iiint 4r\cos\theta\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= 4\,\frac{r^4}{4}\Big|_0^R \cdot \frac{1}{2}\left(-\cos 2\theta\right)\Big|_0^{\pi/2} \cdot \phi\Big|_0^{\pi/2}$$

$$= R^4 . \frac{1}{2}\; -\left(\cos\pi - \cos 0\right) . \frac{\pi}{2}$$

$$= \frac{\pi R^4}{4}$$

$$\oint \bar{v}\cdot d\bar{A} = \iint_1 \bar{v}\cdot \hat{r}\, r^2 \sin\theta\, d\theta\, d\phi + \iint_2 \bar{v}\cdot\hat{\theta}\, dr\, r\sin\theta\, d\phi + \iint_3 \bar{v}\cdot\hat{\phi}\, dr\, r\, d\theta + \iint_4 \bar{v}\cdot(-\hat{\phi})\, dr\, r\, d\theta$$

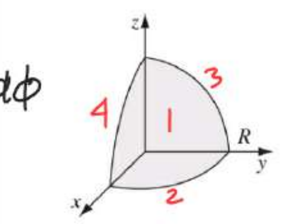


FIGURE 1.48

$$= \iint_1 R^2\cos\theta\, R^2 \sin\theta\, d\theta\, d\phi + \iint_2 r^2\cos\phi\, dr\, r\sin\frac{\pi}{2}\, d\phi + \iint_3 -r^2\cos\theta\sin\frac{\pi}{2}\, dr\, r\, d\theta + \iint_4 r^2\cos\theta\sin 0\, dr\, r\, d\theta$$

$$= R^4 \int_0^{\pi/2}\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta .\, d\phi + \int_0^R r^3\, dr \int_0^{\pi/2}\cos\phi\, d\phi + -\int r^3 dr \int_0^{\pi/2}\cos\theta\, d\theta + 0$$

$$= R^4 \frac{1}{2}\left(-\cos 2\theta\right)\Big|_0^{\pi/2} \phi\Big|_0^{\pi/2}$$

$$= \frac{\pi R^4}{4}$$

<u>Verified</u>

Problem 1.56 Compute the line integral of

$$\mathbf{v} = 6\,\hat{\mathbf{x}} + yz^2\,\hat{\mathbf{y}} + (3y+z)\,\hat{\mathbf{z}}$$

along the triangular path shown in Fig. 1.49. Check your answer using Stokes' theorem. [*Answer:* 8/3]

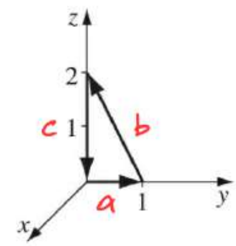


FIGURE 1.49

Solution :

$$\int \bar{v}\cdot d\bar{l} = \int_a \bar{v}\cdot dy\,\hat{y} + \int_b \bar{v}\cdot\left(dy\,\hat{y} + dz\,\hat{z}\right) + \int_c \bar{v}\cdot dz\,\hat{z}$$

$$= \int_a yz^2\,dy + \int_b \left\{ yz^2\,dy + (3y+z)\,dz \right\} + \int_c (3y+z)\,dz$$

along 'b', $z = 2-2y \quad dz = -2dy$

$$= \int_a y\,0^2\cdot dy + \int_1^0 y(2-2y)^2\,dy + (3y+2-2y)(-2dy) + \int_2^0 z\,dz$$

$-2(y+2)$

$$= 0 + \int_1^0 \left(4y - 8y^2 + 4y^3 - 2y - 4\right) dy + \left.\frac{z^2}{2}\right|_2^0$$

$$= \int_1^0 \left(4y^3 - 8y^2 + 2y - 4\right) dy + \left.\frac{z^2}{2}\right|_2^0$$

$$= \left[y^4 - 8\frac{y^3}{3} + y^2 - 4y \right]_1^0 - 2$$

$$= -1 + \frac{8}{3} - 1 + 4 - 2 = 8/3$$

A/C to Stoke's Theorem.

$$\oint \bar{v} \cdot d\bar{l} = \int_s (\bar{\nabla} \times \bar{v}) \cdot d\bar{A}$$

$$\bar{\nabla} \times \bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6 & yz^2 & 3y+z \end{vmatrix}$$

$$= \hat{x}(3 - 2yz) - \hat{y}(0-0) + \hat{z}(0-0)$$

$$= \hat{x}(3 - 2yz)$$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{A} = \iint \hat{x}(3-2yz) \cdot \hat{x}\, dy\, dz$$

$$= \int_0^1 dy \int_0^{2-2y} dz\, (3-2yz)$$

$$= \int_0^1 dy \left(3z - yz^2\right)_0^{2-2y}$$

$$= \int_0^1 dy \left\{ 3(2-2y) - y(2-2y)^2 \right\}$$

$$= \int_0^1 dy \left(6 - 6y - 4y + 8y^2 - 4y^3\right)$$

$$= 6y\Big|_0^1 - 10\frac{y^2}{2}\Big|_0^1 + 8\frac{y^3}{3}\Big|_0^1 - 4\frac{y^4}{4}\Big|_0^1$$

$$= 6 - 5 + \frac{8}{3} - 1$$

$$= 8/3$$

Problem 1.48 Evaluate the following integrals:

(a) $\int (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2)\delta^3(\mathbf{r} - \mathbf{a})\, d\tau$, where $\mathbf{a}$ is a fixed vector, a is its magnitude, and the integral is over all space.

(b) $\int_{\mathcal{V}} |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r})\, d\tau$, where $\mathcal{V}$ is a cube of side 2, centered on the origin, and $\mathbf{b} = 4\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$.

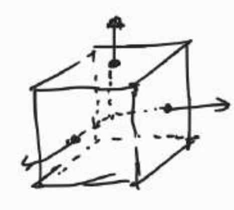
(c) $\int_{\mathcal{V}} \left[r^4 + r^2(\mathbf{r} \cdot \mathbf{c}) + c^4\right] \delta^3(\mathbf{r} - \mathbf{c})\, d\tau$, where $\mathcal{V}$ is a sphere of radius 6 about the origin, $\mathbf{c} = 5\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}$, and c is its magnitude.

(d) $\int_{\mathcal{V}} \mathbf{r} \cdot (\mathbf{d} - \mathbf{r})\delta^3(\mathbf{e} - \mathbf{r})\, d\tau$, where $\mathbf{d} = (1, 2, 3)$, $\mathbf{e} = (3, 2, 1)$, and $\mathcal{V}$ is a sphere of radius 1.5 centered at (2, 2, 2).

a) $\int (r^2 + \bar{r}.\bar{a} + a^2)\,\delta^3(\bar{r}-\bar{a})\,d\tau$

$= a^2 + \bar{a}\cdot\bar{a} + a^2 = 3a^2$

b) $\int |\bar{r}-\bar{b}|^2\, \delta^3(5\bar{r})\, d\tau$



$= \int |\bar{r}-\bar{b}|^2\, \frac{\delta^3(\bar{r})}{5^3}\, d\tau$ **

$= \int |\bar{0}-\bar{b}|^2\, \frac{1}{5^3}$

$= \left|\sqrt{4^2+3^2}\right|^2 \frac{1}{5^3} = \frac{5^2}{5^3} = \frac{1}{5}$

c) $\int_V \left(r^4 + r^2(\bar{r}\cdot\bar{c}) + c^4\right)\delta^3(\bar{r}-\bar{c})$

$c = \sqrt{25+9+4} = \sqrt{38}$

$r = \sqrt{36}$. since $c > r$

$\therefore \int_V \left(r^4 + r^2(\bar{r}\cdot\bar{c}) + c^4\right)\delta^3(\bar{r}-c) = 0$

d) $\int_V \bar{r} \cdot (\bar{d}-\bar{r})\, \delta^3(\bar{e}-\bar{r})\, d\tau$

point is included in the volume

$\bar{e} \cdot (\bar{d}-\bar{e})$

$(3,2,1) \cdot \{(1,2,3)-(3,2,1)\}$

$= (3,2,1) \cdot (-2,0,2)$

$= -6+0+2 = \underline{-4}$

Problem 1.49 Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

(where $\mathcal{V}$ is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.

$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{r}}{r^2} \right) d\tau$

$= \int e^{-r}\, 4\pi\, \delta(r)\, dr$

$= 4\pi e^{-0} = \underline{4\pi}$